

# SUPERCONFORMAL SIMPLE TYPE AND WITTEN'S CONJECTURE

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**ABSTRACT.** Let  $X$  be a smooth, closed, connected, orientable four-manifold with  $b^1(X) = 0$  and  $b^+(X) \geq 3$  and odd. We show that if  $X$  has Seiberg-Witten simple type, then the  $\mathrm{SO}(3)$ -monopole cobordism formula of [6] implies Witten's Conjecture relating the Donaldson and Seiberg-Witten invariants.

## 1. INTRODUCTION

For a closed four-manifold  $X$  we will use the characteristic numbers,

$$(1.1) \quad c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := (e(X) + \sigma(X))/4, \quad c(X) := \chi_h(X) - c_1^2(X),$$

where  $e(X)$  and  $\sigma(X)$  are the Euler characteristic and signature of  $X$ . We call a four-manifold *standard* if it is closed, connected, oriented, and smooth with  $b^+(X) \geq 3$  and odd and  $b^1(X) = 0$ . For a standard four-manifold, the Seiberg-Witten invariants define a function,  $SW_X : \mathrm{Spin}^c(X) \rightarrow \mathbb{Z}$ , on the set of  $\mathrm{spin}^c$  structures on  $X$ . The *Seiberg-Witten basic classes* of  $X$ ,  $B(X)$ , are the image under  $c_1 : \mathrm{Spin}^c(X) \rightarrow H^2(X; \mathbb{Z})$  of the support of  $SW_X$ . The manifold  $X$  has *Seiberg-Witten simple type* if  $K^2 = c_1^2(X)$  for all  $K \in B(X)$ . Further definitions of and notations for the Donaldson and Seiberg-Witten invariants appear in §2.1 and §2.2.

**Conjecture 1.1** (Witten's conjecture). Let  $X$  be a standard four-manifold. If  $X$  has Seiberg-Witten simple type, then  $X$  has Kronheimer-Mrowka simple type, the Seiberg-Witten and Kronheimer-Mrowka basic classes coincide, and for any  $w \in H^2(X; \mathbb{Z})$  and  $h \in H_2(X; \mathbb{R})$  the Donaldson invariants satisfy

$$(1.2) \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s}) \cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h \rangle}.$$

As defined by Mariño, Moore, and Peradze, [31, 30], the manifold  $X$  has *superconformal simple type* if  $c(X) \leq 3$  or  $c(X) \geq 4$  and for  $w \in H^2(X; \mathbb{Z})$  characteristic,

$$(1.3) \quad SW_X^{w,i}(h) := \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s}) \cdot w)} SW_X(\mathfrak{s}) \langle c_1(\mathfrak{s}), h \rangle^i = 0 \quad \text{for } i \leq c(X) - 4,$$

and all  $h \in H_2(X; \mathbb{R})$ . Our goal in this article is to prove the following

**Theorem 1.2.** *Let  $X$  be a standard four-manifold that has superconformal simple type. Then the  $\mathrm{SO}(3)$ -monopole cobordism formula (Theorem 3.2) implies that  $X$  satisfies Witten's Conjecture 1.1.*

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Combining Theorem 1.2 with the results of [7] yields the following

**Corollary 1.3.** *Let  $X$  be a standard four-manifold of Seiberg-Witten simple type and assume Hypothesis 3.1. Then  $X$  satisfies Witten’s Conjecture 1.1.*

In [6], we proved the required  $\mathrm{SO}(3)$ -monopole cobordism formula, restated in this article as Theorem 3.2, assuming the validity of certain technical properties — comprising Hypothesis 3.1 and described in more detail in Remark 3.3 — of the local gluing maps for  $\mathrm{SO}(3)$  monopoles constructed in [9]. A proof of the required local  $\mathrm{SO}(3)$ -monopole gluing-map properties, which may be expected from known properties of local gluing maps for anti-self-dual  $\mathrm{SO}(3)$  connections and Seiberg-Witten monopoles, is currently being developed by the authors [8]. However, Theorem 1.2 is a direct consequence of the  $\mathrm{SO}(3)$ -monopole cobordism formula, Theorem 3.2.

One might draw a comparison between our use of the  $\mathrm{SO}(3)$ -monopole cobordism formula in our proof of Theorem 1.2 and Corollary 1.3 and Göttsche’s assumption of the validity of the Kotschick-Morgan Conjecture [25] in his proof [22] of the wall-crossing formula for Donaldson invariants. However, such a comparison overlooks the fact that our assumption of certain properties for local  $\mathrm{SO}(3)$ -monopole gluing maps is narrower and more specific. Indeed, our monograph [6] effectively contains a proof of the Kotschick-Morgan Conjecture, modulo the assumption of certain technical properties for local gluing maps for anti-self-dual  $\mathrm{SO}(3)$  connections which extend previous results of Taubes [39, 40, 41], Donaldson and Kronheimer [2], and Morgan and Mrowka [34, 35]. Our proof of Theorem 3.2 in [6] relies on our construction of a global gluing map for  $\mathrm{SO}(3)$  monopoles and that in turn builds on properties of local gluing maps for  $\mathrm{SO}(3)$  monopoles; the analogous comments apply to the proof of the Kotschick-Morgan Conjecture.

**1.1. Background.** When defining the Seiberg-Witten invariants in [42], Witten also gave a quantum field theory argument yielding the relation in Conjecture 1.1. Soon after, Pidstrigach and Tyurin [38] introduced the moduli space of  $\mathrm{SO}(3)$  monopoles to give a mathematically rigorous proof of this conjecture. In [6], we used the moduli space of  $\mathrm{SO}(3)$  monopoles to prove — through the assumption of certain properties of local  $\mathrm{SO}(3)$  monopole gluing maps (see [6, Section 6.7] and [10, Remark 3.3]) — the  $\mathrm{SO}(3)$  monopole cobordism formula (Theorem 3.2). This formula gives a relation between the Donaldson and Seiberg-Witten invariants similar to Witten’s Conjecture 1.1, but contains a number of undetermined universal coefficients. In [12, 13] we computed some of these coefficients directly while in [10] we computed more by comparison with known examples. Although these computations showed that Theorem 3.2 implied Conjecture 1.1 for a wide range of standard four-manifolds, they did not suffice for all. In this article, we use the methods of [10] to show that the coefficients not determined in [10, Proposition 4.8] are polynomials in one of the parameters on which they depend. By combining this polynomial dependence with the vanishing condition in the definition of superconformal simple type (1.3), we can show that the sum over the terms in the cobordism formula containing these unknown coefficients vanishes. Hence, the coefficients computed in [10, Proposition 4.8] suffice to determine the Donaldson invariant in terms of Seiberg-Witten invariants and we show that the resulting expression satisfies Conjecture 1.1.

Proofs of Conjecture 1.1 for restricted classes of standard four-manifolds have appeared elsewhere. In [17], Fintushel and Stern proved Conjecture 1.1 for elliptic surfaces and their

blow-ups and rational blow-downs. Kronheimer and Mrowka in [27, Corollary 7] proved that the cobordism formula in Theorem 3.2 implied Conjecture 1.1 for standard four-manifolds with a tight surface with positive self-intersection, a sphere with self-intersection  $(-1)$ , and Euler number and signature equal to that of a smooth hypersurface in  $\mathbb{CP}^3$  of even degree at least six. In [10], we generalized the result of Kronheimer-Mrowka to standard four-manifolds of Seiberg-Witten simple type satisfying  $c(X) \leq 3$  or which are *abundant* in the sense that  $B(X)^\perp \subset H^2(X; \mathbb{Z})$ , the orthogonal complement of the basic classes with respect to the intersection form, contained a hyperbolic summand. (We note that by [11, Section A.2], all simply-connected, closed, complex surfaces with  $b^+ \geq 3$  are abundant.)

T. Mochizuki [32] proved a formula (see Theorem 4.1 in [23]) expressing the Donaldson invariants of a complex projective surface in a form similar to that given by the  $\mathrm{SO}(3)$ -monopole cobordism formula (our Theorem 3.2), but with coefficients given as the residues of an explicit  $\mathbb{C}^*$ -equivariant integral over the product of Hilbert schemes of points on  $X$ . In [23], Göttsche, Nakajima, and Yoshioka express a generating function for these integrals as a meromorphic one-form, given by the “leading terms . . . of Nekrasov’s deformed partition function for the  $N = 2$  SUSY gauge theory with a single fundamental matter” ([23, p. 309]). By extending their meromorphic one-form to  $\mathbb{P}^1$  and analyzing the residues of this form at its poles, the authors of [23] show that all four-manifolds whose Donaldson invariants are given by Mochizuki’s formula satisfy Witten’s Conjecture. This computation implies that the coefficients in Mochizuki’s formula depend on the same data as the coefficients in the  $\mathrm{SO}(3)$ -monopole cobordism formula (see (3.3)) and Göttsche, Nakajima, and Yoshioka conjecture (see [23, Conjecture 4.5]) that Mochizuki’s formula (and thus their proof of Witten’s Conjecture) holds for all standard four-manifolds and not just complex projective surfaces. It is worth noting that the superconformal simple type condition also appears in the proof in [23], specifically [23, Propositions 8.8 and 8.9], but as it is used to analyze the residue of the meromorphic form at one of its poles, superconformal simple type seems to play a role in [23] which is different from that in our article.

The proof in [10] that the  $\mathrm{SO}(3)$  monopole cobordism formula implies Witten’s Conjecture used the result of [4] that abundant four-manifolds have superconformal simple type. In this article, we prove that Theorem 3.2 implies Conjecture 1.1 directly from the superconformal simple type condition. The examples of non-abundant four-manifolds given in [4] (following [20], one takes log transforms on tori in three disjoint nuclei of a K3 surface) show that there are non-abundant four-manifolds which still satisfy the superconformal simple type condition. Hence, the results obtained here are strictly stronger than those in [4].

In [30, 31], Mariño, Moore, and Peradze originally defined the concept of superconformal simple type in the context of supersymmetric quantum field theory and, within that framework, showed that a four-manifold satisfying the superconformal simple type condition obeys the vanishing condition (1.3). They conjectured (see [31, Conjecture 7.8.1]) that all standard four-manifolds of Seiberg-Witten simple type obey (1.3). Not only do all known examples of standard four-manifolds satisfy (1.3) (see [31, Section 7]) but the condition is preserved under the standard surgery operations (blow-up, torus sum, and rational blow-down) used to construct new examples. Using (1.3) as a definition of superconformal simple type, they rigorously derived a lower bound on the number of basic classes for manifolds of superconformal simple type (see [31, Theorem 8.1.1]) in terms of topological invariants of the manifold. Hence, the condition of superconformal simple type is not only of interest

to physicists but has important mathematical implications as evidenced by [31, Theorem 8.1.1], [23, Propositions 8.8], and Theorem 1.2.

Finally, we note that the results of [7] use a variant of the  $\mathrm{SO}(3)$ -monopole cobordism formula to prove that if  $X$  is a standard four-manifold of Seiberg-Witten simple type, then  $X$  has superconformal simple type. Combining this result with Theorem 1.2 gives Corollary 1.3 which completes this part of the  $\mathrm{SO}(3)$ -monopole program.

**1.2. Outline.** After reviewing the definitions of the Seiberg-Witten and Donaldson invariants and the superconformal simple type condition in Section 2, we introduce the  $\mathrm{SO}(3)$ -monopole cobordism formula and some useful reformulations of Conjecture 1.1 in Section 3. The technical heart of the paper appears in Section 4. We cite an algebraic condition, Lemma 4.1, stating when polynomial equations determine coefficients in Section 4.1 and review some basic results on difference operators in Section 4.2. In Section 4.3, we apply Lemma 4.1 to the blow-ups of some examples of standard four-manifolds constructed in [15] which satisfy Conjecture 1.1 to show that the coefficients appearing in the  $\mathrm{SO}(3)$ -monopole cobordism formula are either determined, as in Proposition 4.7, or satisfy a difference equation which determine them up to a polynomial, as in Proposition 4.9. Finally, in Section 5 we prove the crucial Lemma 5.1 which gives a polarized version of the vanishing condition on Seiberg-Witten polynomials appearing in (1.3). Combining Lemma 5.1 with the polynomial dependence of the unknown coefficients shows that the terms with these coefficients can be ignored in the sum giving Donaldson's invariant, thus proving Conjecture 1.1.

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## 2. PRELIMINARIES

We now review the definitions and basic properties of the relevant invariants.

**2.1. Seiberg-Witten invariants.** Detailed expositions of the theory of Seiberg-Witten invariants, introduced by Witten in [42], are provided in [28, 33, 37]. These invariants define an integer-valued map with finite support,

$$SW_X : \mathrm{Spin}^c(X) \rightarrow \mathbb{Z},$$

on the set of  $\mathrm{spin}^c$  structures on  $X$ . A  $\mathrm{spin}^c$  structure,  $\mathfrak{s} = (W^\pm, \rho_W)$  on  $X$ , consists of a pair of complex rank-two bundles  $W^\pm \rightarrow X$  and a Clifford multiplication map  $\rho : T^*X \rightarrow \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$ . If  $\mathfrak{s} \in \mathrm{Spin}^c(X)$ , then  $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X; \mathbb{Z})$  is characteristic.

One calls  $c_1(\mathfrak{s})$  a *Seiberg-Witten basic class* if  $SW_X(\mathfrak{s}) \neq 0$ . Define

$$(2.1) \quad B(X) = \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}.$$

If  $H^2(X; \mathbb{Z})$  has 2-torsion, then  $c_1 : \mathrm{Spin}^c(X) \rightarrow H^2(X; \mathbb{Z})$  is not injective. Because we will work with functions involving real homology and cohomology, we define

$$(2.2) \quad SW'_X : H^2(X; \mathbb{Z}) \rightarrow \mathbb{Z}, \quad SW'_X(K) = \sum_{\mathfrak{s} \in c_1^{-1}(K)} SW_X(\mathfrak{s}).$$

With the preceding definition, Witten's Formula (1.2) is equivalent to

$$(2.3) \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2 + K \cdot w)} SW'_X(K) e^{\langle K, h \rangle}.$$

A four-manifold,  $X$ , has *Seiberg-Witten simple type* if  $SW_X(\mathfrak{s}) \neq 0$  implies that  $c_1^2(\mathfrak{s}) = c_1^2(X)$ .

As discussed in [33, Section 6.8], there is an involution on  $\text{Spin}^c(X)$ , denoted by  $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$  and defined essentially by taking the complex conjugate vector bundles, and having the property that  $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$ . By [33, Corollary 6.8.4], one has  $SW_X(\bar{\mathfrak{s}}) = (-1)^{\chi_h(X)} SW_X(\mathfrak{s})$  and so  $B(X)$  is closed under the action of  $\{\pm 1\}$  on  $H^2(X; \mathbb{Z})$ .

Versions of the following result have appeared in [16], [19, Theorem 14.1.1], and [37, Theorem 4.6.7].

**Theorem 2.1** (Blow-up formula for Seiberg-Witten invariants). [19, Theorem 14.1.1] *Let  $X$  be a standard four-manifold and let  $\tilde{X} = X \# \mathbb{CP}^2$  be its blow-up. Then  $\tilde{X}$  has Seiberg-Witten simple type if and only if that is true for  $X$ . If  $X$  has Seiberg-Witten simple type, then*

$$(2.4) \quad B(\tilde{X}) = \{K \pm e^* : K \in B(X)\},$$

where  $e^* \in H^2(\tilde{X}; \mathbb{Z})$  is the Poincaré dual of the exceptional curve, and if  $K \in B(X)$ , then

$$SW'_{\tilde{X}}(K \pm e^*) = SW'_X(K).$$

**2.2. Donaldson invariants.** In [26, Section 2], Kronheimer and Mrowka defined the Donaldson series which encodes the Donaldson invariants developed in [1]. For  $w \in H^2(X; \mathbb{Z})$ , the *Donaldson invariant* is a linear function,

$$D_X^w : \mathbb{A}(X) \rightarrow \mathbb{R},$$

where  $\mathbb{A}(X)$  is the symmetric algebra,

$$\mathbb{A}(X) = \text{Sym}(H_{\text{even}}(X; \mathbb{R})).$$

For  $h \in H_2(X; \mathbb{R})$  and a generator  $x \in H_0(X; \mathbb{Z})$ , we define  $D_X^w(h^{\delta-2m} x^m) = 0$  unless

$$(2.5) \quad \delta \equiv -w^2 - 3\chi_h(X) \pmod{4}.$$

A four-manifold has *Kronheimer-Mrowka simple type* if for all  $w \in H^2(X; \mathbb{Z})$  and all  $z \in \mathbb{A}(X)$  one has

$$(2.6) \quad D_X^w(x^2 z) = 4D_X^w(z).$$

This equality implies that the Donaldson invariants are determined by the *Donaldson series*, the formal power series

$$(2.7) \quad \mathbf{D}_X^w(h) = D_X^w((1 + \frac{1}{2}x)e^h), \quad h \in H_2(X; \mathbb{R}).$$

The following result allows us to work with a convenient choice of  $w$ :

**Proposition 2.2.** [26], [36, Theorem 2] *Let  $X$  be a standard four-manifold of Seiberg-Witten simple type. If Witten's Conjecture 1.1 holds for one  $w \in H^2(X; \mathbb{Z})$ , then it holds for all  $w \in H^2(X; \mathbb{Z})$ .*

The result below allows us to replace a manifold by its blow-up without loss of generality.

**Theorem 2.3.** [17, Theorem 8.9] *Let  $X$  be a standard four-manifold. Then Witten's Conjecture 1.1 holds for  $X$  if and only if it holds for the blow-up,  $\tilde{X}$ .*

**2.3. Witten's conjecture.** It will be more convenient to have Witten's Conjecture 1.1 expressed at the level of the polynomial invariants rather than the power series they form. Let  $B'(X)$  be a fundamental domain for the action of  $\{\pm 1\}$  on  $B(X)$ .

**Lemma 2.4.** [10, Lemma 4.2] *Let  $X$  be a standard four-manifold. Then  $X$  satisfies equation (1.2) and has Kronheimer-Mrowka simple type if and only if the Donaldson invariants of  $X$  satisfy  $D_X^w(h^{\delta-2m}x^m) = 0$  for  $\delta \not\equiv -w^2 - 3\chi_h \pmod{4}$  and for  $\delta \equiv -w^2 - 3\chi_h \pmod{4}$  satisfy*

$$(2.8) \quad D_X^w(h^{\delta-2m}x^m) = \sum_{\substack{i+2k \\ = \delta-2m}} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} \nu(K) \frac{SW'_X(K)(\delta-2m)!}{2^{k+c(X)-3-m} k! i!} \langle K, h \rangle^i Q_X(h)^k,$$

where

$$(2.9) \quad \varepsilon(w, K) := \frac{1}{2}(w^2 + w \cdot K),$$

and

$$(2.10) \quad \nu(K) = \begin{cases} \frac{1}{2} & \text{if } K = 0, \\ 1 & \text{if } K \neq 0. \end{cases}$$

**2.4. The superconformal simple type property.** A standard four-manifold  $X$  has *superconformal simple type* if  $c(X) \leq 3$  or  $c(X) \geq 4$  and for  $w \in H^2(X; \mathbb{Z})$  characteristic and all  $h \in H_2(X; \mathbb{R})$

$$(2.11) \quad SW_X^{w,i}(h) = \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h \rangle^i = 0 \quad \text{for } i \leq c(X) - 4.$$

Observe that we have rewritten (1.3) as a sum over  $B(X)$  using the expression (2.2). We further note that the property (2.11) is invariant under blow-up.

**Lemma 2.5.** [31, Theorem 7.3.1], [7, Lemma 6.1] *A standard manifold,  $X$ , has superconformal simple type if and only if its blow-up,  $\tilde{X}$ , has superconformal simple type.*

### 3. $\text{SO}(3)$ MONOPOLES AND WITTEN'S CONJECTURE

The  $\text{SO}(3)$ -monopole cobordism formula (3.2) given in Theorem 3.2 provides an expression for the Donaldson invariant in terms of the Seiberg-Witten invariants.

*Hypothesis 3.1* (Properties of local  $\text{SO}(3)$ -monopole gluing maps). The local gluing map, constructed in [9], gives a continuous parametrization of a neighborhood of  $M_s \times \Sigma$  in  $\bar{\mathcal{M}}_t$  for each smooth stratum  $\Sigma \subset \text{Sym}^\ell(X)$ .

Hypothesis 3.1 is discussed in greater detail in [6, Section 6.7]. The question of how to assemble the *local* gluing maps for neighborhoods of  $M_{\mathfrak{s}} \times \Sigma$  in  $\bar{\mathcal{M}}_{\mathfrak{t}}$ , as  $\Sigma$  ranges over all smooth strata of  $\text{Sym}^{\ell}(X)$ , into a *global* gluing map for a neighborhood of  $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$  in  $\bar{\mathcal{M}}_{\mathfrak{t}}$  is itself difficult — involving the so-called ‘overlap problem’ described in [14] — but one which we do solve in [6]. See Remark 3.3 for a further discussion of this point.

**Theorem 3.2** (SO(3)-monopole cobordism formula). [6] *Let  $X$  be a standard four-manifold of Seiberg-Witten simple type and assume Hypothesis 3.1. Assume further that  $w, \Lambda \in H^2(X; \mathbb{Z})$  and  $\delta, m \in \mathbb{N}$  satisfy*

$$(3.1a) \quad w - \Lambda \equiv w_2(X) \pmod{2},$$

$$(3.1b) \quad I(\Lambda) = \Lambda^2 + c(X) + 4\chi_h(X) > \delta,$$

$$(3.1c) \quad \delta \equiv -w^2 - 3\chi_h(X) \pmod{4},$$

$$(3.1d) \quad \delta - 2m \geq 0.$$

Then, for any  $h \in H_2(X; \mathbb{R})$  and positive generator  $x \in H_0(X; \mathbb{Z})$ , we have

$$(3.2) \quad \begin{aligned} & D_X^w(h^{\delta-2m} x^m) \\ &= \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2 - \sigma) + \frac{1}{2}(w^2 + (w - \Lambda) \cdot K)} SW'_X(K) f_{\delta, m}(\chi_h(X), c_1^2(X), K, \Lambda)(h), \end{aligned}$$

where the map,

$$f_{\delta, m}(h) : \mathbb{Z} \times \mathbb{Z} \times H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \rightarrow \mathbb{R}[h],$$

taking values in the ring of polynomials in the variable  $h$  with real coefficients, is universal (independent of  $X$ ) and given by

$$(3.3) \quad \begin{aligned} & f_{\delta, m}(\chi_h(X), c_1^2(X), K, \Lambda)(h) \\ &:= \sum_{\substack{i+j+2k \\ = \delta - 2m}} a_{i, j, k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m) \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k, \end{aligned}$$

and, for each triple of non-negative integers,  $i, j, k \in \mathbb{N}$ , the coefficients,

$$a_{i, j, k} : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{R},$$

are real analytic (independent of  $X$ ) functions of the variables  $\chi_h(X)$ ,  $c_1^2(X)$ ,  $c_1(\mathfrak{s}) \cdot \Lambda$ ,  $\Lambda^2$ , and  $m$ .

**Remark 3.3.** The proof of Theorem 3.2 in [6] assumes the Hypothesis 3.1 (see [6, Section 6.7]) that the local gluing map for a neighborhood of  $M_{\mathfrak{s}} \times \Sigma$  in  $\bar{\mathcal{M}}_{\mathfrak{t}}$  gives a continuous parametrization of a neighborhood of  $M_{\mathfrak{s}} \times \Sigma$  in  $\bar{\mathcal{M}}_{\mathfrak{t}}$ , for each smooth stratum  $\Sigma \subset \text{Sym}^{\ell}(X)$ . These local gluing maps are the analogues for SO(3) monopoles of the local gluing maps for anti-self-dual SO(3) connections constructed by Taubes in [39, 40, 41] and Donaldson and Kronheimer in [2, §7.2]; see also [34, 35]. We have established the existence of local gluing maps in [9] and expect that a proof of the continuity for the local gluing maps with respect to Uhlenbeck limits should be similar to our proof in [5] of this property for the local gluing maps for anti-self-dual SO(3) connections. The remaining properties of local gluing maps assumed in [6] are that they are injective and also surjective in the sense that elements of  $\bar{\mathcal{M}}_{\mathfrak{t}}$  sufficiently close (in the Uhlenbeck topology) to  $M_{\mathfrak{s}} \times \Sigma$  are in the image of at least

one of the local gluing maps. In special cases, proofs of these properties for the local gluing maps for anti-self-dual  $\mathrm{SO}(3)$  connections (namely, continuity with respect to Uhlenbeck limits, injectivity, and surjectivity) have been given in [2, §7.2.5, 7.2.6], [39, 40, 41]. The authors are currently developing a proof of the required properties for the local gluing maps for  $\mathrm{SO}(3)$  monopoles [8]. Our proof will also yield the analogous properties for the local gluing maps for anti-self-dual  $\mathrm{SO}(3)$  connections, as required to complete the proof of the Kotschick-Morgan Conjecture [25], based on our work in [6].

It will be convenient for us to rewrite Theorem 3.2 as a sum over  $B'(X) \subset B(X)$ , a fundamental domain for the action of  $\{\pm 1\}$ , to compare with Lemma 2.4. To this end, we follow [10, Equation (4.4)] and define

$$\begin{aligned} b_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m) \\ := (-1)^{c(X)+i} a_{i,j,k}(\chi_h(X), c_1^2(X), -K \cdot \Lambda, \Lambda^2, m) \\ + a_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m), \end{aligned}$$

where  $a_{i,j,k}$  are the coefficients appearing in (3.3). To simplify the orientation factor in (3.2), we define

$$\begin{aligned} (3.4) \quad \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m) \\ := (-1)^{\frac{1}{2}(\Lambda^2 + \Lambda \cdot K)} b_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m). \end{aligned}$$

Observe that

$$\begin{aligned} (3.5) \quad \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), -K \cdot \Lambda, \Lambda^2, m) \\ = (-1)^{c(X)+i+\Lambda \cdot K} \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m). \end{aligned}$$

We now rewrite (3.2) as a sum over  $B'(X)$ .

**Lemma 3.4.** *Assume the hypotheses of Theorem 3.2. Denote the coefficients in (3.5) more concisely by*

$$\tilde{b}_{i,j,k}(K \cdot \Lambda) := \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m).$$

*Then, for  $\varepsilon(w, K) = \frac{1}{2}(w^2 + w \cdot K)$  as in (2.9),*

$$\begin{aligned} (3.6) \quad D_X^w(h^{\delta-2m} x^m) &= \sum_{K \in B'(X)} \sum_{\substack{i+j+2k \\ = \delta-2m}} \nu(K) (-1)^{\varepsilon(w, K)} SW'_X(K) \\ &\quad \times \tilde{b}_{i,j,k}(K \cdot \Lambda) \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k, \end{aligned}$$

where  $\nu(K)$  is defined by (2.10).

*Proof.* We first compare the orientation factors of  $\varepsilon(w, K)$  appearing in (2.8) and  $\frac{1}{2}(w^2 - \sigma) + \frac{1}{2}(w^2 + (w - \Lambda) \cdot K)$  appearing in (3.2). Because  $w - \Lambda$  is characteristic by (3.1a), we have

$$(3.7a) \quad \sigma(X) \equiv (w - \Lambda)^2 \pmod{8} \quad (\text{by [21, Lemma 1.2.20]}),$$

$$(3.7b) \quad \Lambda \cdot (w - \Lambda) \equiv \Lambda^2 \pmod{2}.$$



Then,

$$\begin{aligned}
 & \frac{1}{2}(w^2 - \sigma(X)) + \frac{1}{2}(w^2 + (w - \Lambda) \cdot K) \\
 & \equiv \varepsilon(w, K) + \frac{1}{2}(w^2 - \sigma(X)) - \frac{1}{2}\Lambda \cdot K \pmod{2} \quad (\text{by (2.9)}) \\
 & \equiv \varepsilon(w, K) + \frac{1}{2}(w^2 - (w - \Lambda)^2 - \Lambda \cdot K) \pmod{2} \quad (\text{by (3.7a)}) \\
 & \equiv \varepsilon(w, K) + \frac{1}{2}(2w \cdot \Lambda - \Lambda^2 - \Lambda \cdot K) \pmod{2} \\
 & \equiv \varepsilon(w, K) + \frac{1}{2}(2w \cdot \Lambda - 2\Lambda^2 + \Lambda^2 - \Lambda \cdot K) \pmod{2} \\
 & \equiv \varepsilon(w, K) - (\Lambda - w) \cdot \Lambda + \frac{1}{2}(\Lambda^2 - \Lambda \cdot K) \pmod{2} \\
 & \equiv \varepsilon(w, K) - \Lambda^2 + \frac{1}{2}(\Lambda^2 - \Lambda \cdot K) \pmod{2} \quad (\text{by (3.7b)}),
 \end{aligned}$$

and hence,

$$(3.8) \quad \frac{1}{2}(w^2 - \sigma(X)) + \frac{1}{2}(w^2 + (w - \Lambda) \cdot K) \equiv \varepsilon(w, K) - \frac{1}{2}(\Lambda^2 + \Lambda \cdot K) \pmod{2}.$$

From [10, Lemma 4.3], we have

$$\begin{aligned}
 (3.9) \quad D_X^w(h^{\delta-2m}x^m) &= \sum_{K \in B'(X)} \sum_{\substack{i+j+2k \\ = \delta-2m}} \nu(K)(-1)^{\frac{1}{2}(w^2 - \sigma(X)) + \frac{1}{2}(w^2 + (w - \Lambda) \cdot K)} SW'_X(K) \\
 &\quad \times b_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m) \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k,
 \end{aligned}$$

The result (3.6) now follows from (3.8), (3.9), and the relation between the coefficients  $\tilde{b}_{i,j,k}$  and  $b_{i,j,k}$  in (3.4).  $\square$

The following lemma allows us to ignore the coefficients  $\tilde{b}_{0,j,k}$  for the purpose of proving Theorem 1.2 and Corollary 1.3.

**Lemma 3.5.** *Continue the notation and hypotheses of Lemma 3.4. Then,*

$$\begin{aligned}
 (3.10) \quad D_X^w(h^{\delta-2m}x^m) &= \sum_{K \in B'(X)} \sum_{\substack{i+j+2k \\ = \delta-2m}} (-1)^{\varepsilon(w, K)} SW'_X(K) \frac{2(i+1)}{(\delta-2m+1)} \\
 &\quad \times \tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) \\
 &\quad \times \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k.
 \end{aligned}$$

*Proof.* Let  $\tilde{X} \rightarrow X$  be the blow-up of  $X$  at one point, let  $e \in H_2(\tilde{X}; \mathbb{Z})$  be the fundamental class of the exceptional curve, and let  $e^* \in H^2(\tilde{X}; \mathbb{Z})$  be the Poincaré dual of  $e$ . Using the direct sum decomposition of the homology and cohomology of  $\tilde{X}$ , we will consider both the homology and cohomology of  $X$  as subspaces of those of  $\tilde{X}$ . Denote  $\tilde{w} := w + e^*$ . The blow-up formula [24, 29] gives

$$(3.11) \quad D_X^w(h^{\delta-2m}x^m) = D_{\tilde{X}}^{\tilde{w}}(h^{\delta-2m}ex^m).$$

By Theorem 2.1,

$$(3.12) \quad B'(\tilde{X}) = \{K_\varphi = K + (-1)^\varphi e^* : K \in B'(X), \varphi \in \mathbb{Z}/2\mathbb{Z}\}.$$

To apply the cobordism formula (3.6) to compute the right-hand-side of (3.11), we must discuss the isomorphism,  $\Phi$ , from the space of symmetric,  $d$ -linear functionals on a real vector space,  $V$ , onto the space of degree- $d$  polynomials on  $V$ , defined by (see [18, Section 6.1.1])

$$\Phi(M)(h) := M(\underbrace{h, \dots, h}_{d \text{ copies}}).$$

If  $F = \Phi(M)$  is a degree  $d$ -polynomial, we can find  $M$  by the formula, [18, p. 396],

$$(3.13) \quad M(h_1, \dots, h_d) = \frac{1}{d!} \frac{\partial^d}{\partial t_1 \partial t_2 \dots \partial t_d} F(t_1 h_1 + \dots + t_d h_d) \Big|_{t_1 = \dots = t_d = 0}.$$

For the polynomial of degree  $\delta - 2m + 1 = i + j + 2k$  defined by

$$F_{i,j,k}^\varphi(\tilde{h}) := \langle K_\varphi, \tilde{h} \rangle^i \langle \Lambda, \tilde{h} \rangle^j Q_{\tilde{X}}(\tilde{h})^k, \quad \forall \tilde{h} \in H_2(\tilde{X}; \mathbb{R}),$$

where (as usual)  $\Lambda \in H^2(X; \mathbb{Z})$ , the identity (3.13) implies that the functional  $M_{i,j,k}^\varphi := \Phi^{-1}(F_{i,j,k}^\varphi)$  satisfies

$$(3.14) \quad M_{i,j,k}^\varphi(e, h, \dots, h) = \frac{i(-1)^{\varphi+1}}{(\delta - 2m + 1)} \langle K, h \rangle^{i-1} \langle \Lambda, h \rangle^j Q_X(h)^k, \quad \forall h \in H_2(X; \mathbb{R}).$$

Before applying the cobordism formula to the right-hand-side of (3.11), we check that the conditions (3.1) of Theorem 3.2 hold.

Our assumption that  $\tilde{w} = w + e^*$  ensures that for  $\Lambda \in H^2(X; \mathbb{Z})$ , we have  $\Lambda + \tilde{w} \equiv w_2(\tilde{X})$  if and only if  $\Lambda + w \equiv w_2(X)$ . Hence, the condition (3.1a) holds for  $w$  and  $\Lambda$  on  $X$  if and only if it holds for  $\tilde{w}$  and  $\Lambda$  on  $\tilde{X}$ .

Because  $c(\tilde{X}) = c(X) + 1$  and  $\chi_h(\tilde{X}) = \chi_h(X)$ , we see that  $\Lambda^2 + c(X) + 4\chi_h(X) > \delta$  if and only if  $\Lambda^2 + c(\tilde{X}) + 4\chi_h(\tilde{X}) > \delta + 1$ . Consequently, the condition (3.1b) holds for  $\Lambda$ ,  $\delta$  and  $X$  if and only if it holds for  $\Lambda$ ,  $\delta + 1$ , and  $\tilde{X}$ .

Since  $-\tilde{w}^2 = -w^2 + 1$ , we have  $\delta + 1 \equiv \tilde{w}^2 - 3\chi_h(\tilde{X}) \pmod{4}$  if and only if  $\delta \equiv \tilde{w}^2 - 3\chi_h(X) \pmod{4}$ . Therefore, the condition (3.1c) holds for  $w$ ,  $\delta$  and  $X$  if and only if it holds for  $\tilde{w}$ ,  $\delta + 1$ , and  $\tilde{X}$ .

Finally, if the condition (3.1d) holds for  $\delta$  and  $m$ , then it holds for  $\delta + 1$  and  $m$ .

Thus, if we assume that the conditions (3.1) in Theorem 3.2 hold for  $w$ ,  $\Lambda$ ,  $\delta$ , and  $m$  on  $X$ , then they hold for  $\tilde{w}$ ,  $\Lambda$ ,  $\delta + 1$ , and  $m$  on  $\tilde{X}$ . Hence, we can apply (3.6) in Lemma 3.4 to compute the right-hand-side of (3.11). Note that for any  $K \in B'(X)$ ,

$$\varepsilon(w, K) \equiv \varepsilon(\tilde{w}, K_1) \equiv \varepsilon(\tilde{w}, K_0) + 1 \pmod{2},$$

where  $K_0, K_1 \in B'(\tilde{X})$  as in (3.12). Also, because  $\Lambda \in H^2(X; \mathbb{Z})$ ,

$$\tilde{b}_{i,j,k}(\chi_h(\tilde{X}), c_1^2(\tilde{X}), K_\varphi \cdot \Lambda, \Lambda^2, m) = \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m).$$

Applying the definition of  $M_{i,j,k}^\varphi$  from (3.14) to the cobordism formula (3.6) for  $\Lambda$ ,  $\tilde{w}$ , and  $\delta + 1$  on  $\tilde{X}$  then gives us

$$\begin{aligned}
 & D_{\tilde{X}}^{\tilde{w}}(h^{\delta-2m} ex^m) \\
 &= \sum_{K_\varphi \in B'(\tilde{X})} \sum_{\substack{i+j+2k \\ =\delta-2m+1}} (-1)^{\varepsilon(w, K_\varphi)} SW_{\tilde{X}}'(K_\varphi) \\
 &\quad \times \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) M_{i,j,k}^\varphi(e, h, \dots, h) \\
 &= \sum_{K \in B'(X)} \sum_{\substack{i+j+2k \\ =\delta-2m+1}} (-1)^{\varepsilon(w, K)} SW_X(K) \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) \\
 &\quad \times (M_{i,j,k}^1(e, h, \dots, h) - M_{i,j,k}^0(e, h, \dots, h)) \\
 &= \sum_{K \in B'(X)} \sum_{\substack{i+j+2k \\ =\delta-2m+1}} (-1)^{\varepsilon(w, K)} SW_X(K) \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) \\
 &\quad \times \frac{2i}{(\delta - 2m + 1)} \langle K, h \rangle^{i-1} \langle \Lambda, h \rangle^j Q_X(h)^k \quad (\text{by (3.14)}) \\
 &= \sum_{K \in B'(X)} \sum_{\substack{i+j+2k \\ =\delta-2m}} (-1)^{\varepsilon(w, K)} SW_X(K) \tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) \\
 &\quad \times \frac{2(i+1)}{(\delta - 2m + 1)} \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k.
 \end{aligned}$$

and thus,

$$\begin{aligned}
 & D_{\tilde{X}}^{\tilde{w}}(h^{\delta-2m} ex^m) \\
 &= \sum_{K \in B'(X)} \sum_{\substack{i+j+2k \\ =\delta-2m}} (-1)^{\varepsilon(w, K)} SW_X(K) \tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) \\
 &\quad \times \frac{2(i+1)}{(\delta - 2m + 1)} \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k.
 \end{aligned}$$

Combining the preceding identity and (3.11) gives the result.  $\square$

#### 4. CONSTRAINING THE COEFFICIENTS

In this section, we show that the coefficients  $\tilde{b}_{i,j,k}$  appearing in (3.6) which are not determined by [10, Proposition 4.8] satisfy a difference equation in the parameter  $K \cdot \Lambda$  and thus can be written as a polynomial in this parameter.

**4.1. Algebraic preliminaries.** To determine the coefficients  $\tilde{b}_{i,j,k}$  appearing in (3.6), we compare equations (2.8) and (3.6) on manifolds where Witten's Conjecture 1.1 is known to hold and use the following generalization of [18, Lemma VI.2.4].

**Lemma 4.1.** [10, Lemma 4.1] *Let  $V$  be a finite-dimensional real vector space. Let  $T_1, \dots, T_n$  be linearly independent elements of the dual space  $V^*$ . Let  $Q$  be a quadratic form on  $V$  which is non-zero on  $\cap_{i=1}^n \text{Ker } T_i$ . Then  $T_1, \dots, T_n, Q$  are algebraically independent in the sense*

that if  $F(z_0, \dots, z_n) \in \mathbb{R}[z_0, \dots, z_n]$  and  $F(Q, T_1, \dots, T_n) : V \rightarrow \mathbb{R}$  is the zero map, then  $F(z_0, \dots, z_n)$  is the zero element of  $\mathbb{R}[z_0, \dots, z_n]$ .

**4.2. Difference equations.** We review some notation and results for difference operators. For  $f : \mathbb{Z} \rightarrow \mathbb{R}$  and  $p, q \in \mathbb{Z}$ , define

$$(\nabla_p^q f)(x) := f(x) + (-1)^q f(x + p).$$

For  $a \in \mathbb{Z}/2\mathbb{Z}$  and  $p \in \mathbb{Z}$ , define  $pa, ap \in \mathbb{Z}$  by

$$(4.1) \quad pa = ap = -\frac{1}{2}(-1 + (-1)^a)p = \begin{cases} 0 & \text{if } a \equiv 0 \pmod{2}, \\ p & \text{if } a \equiv 1 \pmod{2}. \end{cases}$$

We recall the

**Lemma 4.2.** [10, Lemma 4.6] *For all  $(p_1, \dots, p_n)$  and  $(q_1, \dots, q_n) \in \mathbb{Z}^n$ , there holds*

$$\sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n q_u \pi_u(\varphi)} f\left(x + \sum_{u=1}^n p_u \pi_u(\varphi)\right) = (\nabla_{p_1}^{q_1} \nabla_{p_2}^{q_2} \dots \nabla_{p_n}^{q_n} f)(x),$$

where  $\pi_u : (\mathbb{Z}/2\mathbb{Z})^n \rightarrow \mathbb{Z}/2\mathbb{Z}$  is projection onto the  $u$ -th factor and, for a constant function,  $C$ , there holds

$$(4.2) \quad (\nabla_{p_n}^{q_n} \nabla_{p_{n-1}}^{q_{n-1}} \dots \nabla_{p_1}^{q_1} C) = \begin{cases} 0, & \text{if } \exists u \text{ with } 1 \leq u \leq n \text{ and } q_u \equiv 1 \pmod{2}, \\ 2^n C, & \text{if } q_u \equiv 0 \pmod{2} \forall u \text{ with } 1 \leq u \leq n. \end{cases}$$

We will also use the following similar result (compare [3, Lemma 2.22]).

**Lemma 4.3.** *For  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  and  $\lambda \in \mathbb{Z}$ , there holds*

$$((\nabla_\lambda^1)^n f)(x) = \sum_{i=0}^n (-1)^i \binom{n}{i} f(x + i\lambda).$$

*Proof.* If  $E_\lambda$  is the translation operator,  $E_\lambda f(x) = f(x + \lambda)$ , and  $I$  is the identity, then  $\nabla_\lambda^1 = I - E_\lambda$ . The lemma then follows from a binomial expansion.  $\square$

We add the following

**Lemma 4.4.** *Let  $\lambda \in \mathbb{Z}$  and  $p : \mathbb{Z} \rightarrow \mathbb{R}$  be a function.*

- (1) *If  $\nabla_\lambda^1 p(x)$  is a polynomial of degree  $n$  in  $x$ , then  $p(\lambda x)$  is a polynomial of degree  $n + 1$ ;*
- (2) *If  $\nabla_\lambda^1 p(x) = 0$ , then  $p(\lambda x)$  is constant.*

*Proof.* Note that the lemma is trivial if  $\lambda = 0$ . The second statement follows trivially from the definitions.

We prove the first statement by induction on  $n$ . If  $n = 0$ , then there is a constant  $C_1$  such that  $p(x) - p(x + \lambda) = C_1$  for all  $x$  and hence  $p(\lambda x) = -C_1 x + C_2$ , where  $C_2 = p(0)$ .

For the inductive step, assume that  $\nabla_\lambda^1 p(x)$  is a polynomial of degree  $m$  and define  $q(x) := p(\lambda x)$ . Because  $(\nabla_1^1 q)(x) = (\nabla_\lambda^1 p)(\lambda x)$ , we see that  $(\nabla_1^1 q)(x) = Cx^m + r(x)$ , where  $r(x)$  is a polynomial of degree  $m - 1$ . We compute that

$$\nabla_1^1 \left( q(x) + \frac{C}{m+1} x^{m+1} \right)$$

is a polynomial of degree  $m-1$  and so, by induction,  $q(x) + Cx^{m+1}/(m+1)$  is a polynomial of degree  $m$ . Hence,  $q(x) = p(\lambda x)$  is a polynomial of degree  $m+1$ , completing the induction.  $\square$

**Corollary 4.5.** *For  $\lambda \neq 0$ , let  $c : \mathbb{Z} \rightarrow \mathbb{R}$  be a function satisfying,*

$$\underbrace{(\nabla_\lambda^1 \nabla_\lambda^1 \cdots \nabla_\lambda^1 c)}_{n \text{ copies}}(\lambda x) = 0,$$

*for all  $x \in \mathbb{Z}$ . Then  $c_\lambda(x) = c(\lambda x)$  is a polynomial in  $x$  of degree  $n-1$ .*

*Proof.* From Lemma 4.3, one can see that  $c_\lambda$  satisfies  $(\nabla_1^1 \cdots \nabla_1^1 c_\lambda)(x) = 0$ . The result then follows from Lemma 4.4 and induction on  $n$ .  $\square$

**4.3. The example manifolds and blow-up formulas.** In [10, Section 4.2], we used the manifolds constructed by Fintushel, Park and Stern in [15] to give a family of standard four-manifolds,  $X_q$ , for  $q = 2, 3, \dots$ , obeying the following conditions:

- (1)  $X_q$  satisfies Witten's Conjecture 1.1;
- (2) For  $q = 2, 3, \dots$ , one has  $\chi_h(X_q) = q$  and  $c(X_q) = 3$ ;
- (3)  $B'(X_q) = \{K\}$  with  $K \neq 0$ ;
- (4) For each  $q$ , there are classes  $f_1, f_2 \in H^2(X_q; \mathbb{Z})$  satisfying

$$(4.3a) \quad f_1 \cdot f_2 = 1 \quad \text{and} \quad f_i^2 = 0 \quad \text{and} \quad f_i \cdot K = 0 \quad \text{for} \quad i = 1, 2,$$

$$(4.3b) \quad \text{The cohomology classes } \{f_1, f_2, K\} \text{ are linearly independent in } H^2(X_q; \mathbb{R}),$$

$$(4.3c) \quad \text{The restriction of } Q_{X_q} \text{ to } \text{Ker } f_1 \cap \text{Ker } f_2 \cap \text{Ker } K \text{ is non-zero.}$$

Let  $X_q(n)$  be the blow-up of  $X_q$  at  $n$  points,

$$(4.4) \quad X_q(n) := X_q \underbrace{\# \overline{\mathbb{CP}}^2 \cdots \# \overline{\mathbb{CP}}^2}_{n \text{ copies}}.$$

Then  $X_q(n)$  is a standard four-manifold of Seiberg-Witten simple type and satisfies Witten's Conjecture 1.1 by Theorem 2.3, with

$$(4.5) \quad \chi_h(X_q(n)) = q, \quad c_1^2(X_q(n)) = q - n - 3, \quad \text{and} \quad c(X_q(n)) = n + 3.$$

We will consider both the homology and cohomology of  $X_q$  as subspaces of those of  $X_q(n)$ . Let  $e_u^* \in H^2(X_q(n); \mathbb{Z})$  be the Poincaré dual of the  $u$ -th exceptional class. Let  $\pi_u : (\mathbb{Z}/2\mathbb{Z})^n \rightarrow \mathbb{Z}/2\mathbb{Z}$  be projection onto the  $u$ -th factor. For  $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$ , we define

$$(4.6) \quad K_\varphi := K + \sum_{u=1}^n (-1)^{\pi_u(\varphi)} e_u^* \quad \text{and} \quad K_0 := K + \sum_{u=1}^n e_u^*.$$

By Theorem 2.1,

$$(4.7) \quad B'(X_q(n)) = \{K_\varphi : \varphi \in (\mathbb{Z}/2\mathbb{Z})^n\},$$

and, for all  $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$ ,

$$(4.8) \quad SW'_{X_q(n)}(K_\varphi) = SW'_{X_q}(K).$$

Because  $X_q(n)$  has Seiberg-Witten simple type, we have

$$(4.9) \quad K_\varphi^2 = c_1^2(X_q(n)) \quad \text{for all } \varphi \in (\mathbb{Z}/2\mathbb{Z})^n.$$

In addition, because  $K \neq 0$ , we see that

$$(4.10) \quad 0 \notin B'(X_q(n)).$$

Noting that the manifolds  $X_q(n)$  satisfy Witten's Conjecture 1.1, Lemma 4.1 and the equality given by combining equations (2.8) and (3.6), applied to the manifolds  $X_q(n)$ , will show that the coefficients  $\tilde{b}_{i,j,k}$  satisfy certain difference equations. Those difference equations will allow us to prove Theorem 1.2.

For  $n \geq 2$ , the set  $B'(X_q(n))$  is not linearly independent in  $H^2(X_q(n); \mathbb{R})$ . To apply Lemma 4.1, we need to replace  $B'(X_q(n))$  with a linearly independent set. To this end, we give the following formula for the Donaldson invariants of  $X_q(n)$ . It differs from [10, Lemma 4.7] in the change of coefficients from  $b_{i,j,k}$  to  $\tilde{b}_{i,j,k}$  and in our use of the linearly independent set  $K \pm e_1^*, e_2^*, \dots, e_n^*$ .

**Lemma 4.6.** *For  $n, q \in \mathbb{Z}$  with  $n \geq 1$  and  $q \geq 2$ , let  $X_q(n)$  be the manifold defined in (4.4). For  $\Lambda, w \in H^2(X_q; \mathbb{Z})$  and  $\delta, m \in \mathbb{N}$  satisfying  $\Lambda - w \equiv w_2(X_q) \pmod{2}$  and  $\delta - 2m \geq 0$ , define  $\tilde{w}, \tilde{\Lambda} \in H^2(X_q(n); \mathbb{Z})$  by*

$$(4.11) \quad \tilde{w} := w + \sum_{u=1}^n w_u e_u^* \quad \text{and} \quad \tilde{\Lambda} := \Lambda + \sum_{u=1}^n \lambda_u e_u^*,$$

where  $w_u, \lambda_u \in \mathbb{Z}$  and  $w_u + \lambda_u \equiv 1 \pmod{2}$  for  $u = 1, \dots, n$ . We assume that

$$(4.12a) \quad \Lambda^2 > \delta - (n+3) - 4q + \sum_{u=1}^n \lambda_u^2,$$

$$(4.12b) \quad \delta \equiv -w^2 + \sum_{u=1}^n w_u^2 - 3q \pmod{4}.$$

Denote  $x := \tilde{K}_\varphi \cdot \tilde{\Lambda}$  and, for  $i, j, k \in \mathbb{N}$  satisfying  $i + j + 2k + 2m = \delta$ , write

$$\tilde{b}_{i,j,k}(x) = \tilde{b}_{i,j,k}(\chi_h(X_q(n)), c_1^2(X_q(n)), x, \tilde{\Lambda}^2, m).$$

Then, for  $x_0 = K_0 \cdot \tilde{\Lambda}$  where  $K_0$  is defined in (4.6),

$$(4.13) \quad \begin{aligned} & \sum_{\substack{i_1 + \dots + i_n + 2k \\ = \delta - 2m}} \frac{(\delta - 2m)!}{2^{k+n-m} k! i_1! \dots i_n!} p^{\tilde{w}}(i_2, \dots, i_n) \left( \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \right) Q_{X_q(n)}(h)^k \\ & \quad \times (\langle K + e_1^*, h \rangle^{i_1} + (-1)^{w_1} \langle K - e_1^*, h \rangle^{i_1}) \\ & = \sum_{\substack{i_1 + \dots + i_n + j + 2k \\ = \delta - 2m}} \binom{i_1 + \dots + i_n}{i_1, \dots, i_n} \langle \tilde{\Lambda}, h \rangle^j \left( \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \right) Q_{X_q(n)}(h)^k \\ & \quad \times \left( \nabla_{2\lambda_2}^{i_2 + w_2} \dots \nabla_{2\lambda_n}^{i_n + w_n} \tilde{b}_{i,j,k}(x_0) \langle K + e_1^*, h \rangle^{i_1} \right. \\ & \quad \left. + (-1)^{w_1} \nabla_{2\lambda_2}^{i_2 + w_2} \dots \nabla_{2\lambda_n}^{i_n + w_n} \tilde{b}_{i,j,k}(x_0 + 2\lambda_1) \langle K - e_1^*, h \rangle^{i_1} \right), \end{aligned}$$

where  $\tilde{\Lambda}$  is as defined in (4.11) and

$$(4.14) \quad p^{\tilde{w}}(i_2, \dots, i_n) = \begin{cases} 0 & \text{if } \exists u \text{ with } 2 \leq u \leq n \text{ and } w_u + i_u \equiv 1 \pmod{2}, \\ 2^{n-1} & \text{if } w_u + i_u \equiv 0 \pmod{2} \forall u \text{ with } 2 \leq u \leq n. \end{cases}$$

*Proof.* We first verify that  $\tilde{\Lambda}$ ,  $\tilde{w}$ ,  $\delta$ , and  $m$  satisfy the hypotheses (3.1) in Theorem 3.2 for the manifold  $X_q(n)$ .

Because  $\Lambda - w \equiv w_2(X_q)$  and  $\lambda_u + w_u \equiv 1 \pmod{2}$ , the definition (4.11) of  $\tilde{\Lambda}$  and  $\tilde{w}$  and the equality  $w_2(X_q(n)) \equiv w_2(X_q) + \sum_{u=1}^n e_u^* \pmod{2}$  imply that  $\tilde{\Lambda}$  and  $\tilde{w}$  satisfy the condition (3.1a) for  $X_q(n)$ .

The definition (4.11) of  $\tilde{\Lambda}$  also implies that  $\tilde{\Lambda}^2 = \Lambda^2 - \sum_{u=1}^n \lambda_u^2$ . Together with (4.12a) and the equalities  $c(X_q(n)) = n + 3$  and  $\chi_h(X_q(n)) = q$  from (4.5), this yields

$$\tilde{\Lambda}^2 = \Lambda^2 - \sum_{u=1}^n \lambda_u^2 > \delta - c(X_q(n)) - 4\chi_h(X_q(n)),$$

so  $\tilde{\Lambda}$  and  $\delta$  satisfy (3.1b) on  $X_q(n)$ .

The definition of  $\tilde{w}$  gives  $-\tilde{w}^2 = -w^2 + \sum_{u=1}^n w_u^2$ . Combining this equality with the assumption (4.12b) and the equality  $\chi_h(X_q(n)) = q$  from (4.5), we obtain

$$\delta \equiv -\tilde{w}^2 - 3\chi_h(X_q(n)) \pmod{4},$$

so  $\delta$  and  $\tilde{w}$  satisfy (3.1c) on  $X_q(n)$ .

The condition (3.1d) appears directly as the hypothesis  $\delta \geq 2m$  in Lemma 4.6. Hence, we can apply Theorem 3.2 with  $\tilde{\Lambda}$ ,  $\tilde{w}$ ,  $\delta$ , and  $m$  for the manifold  $X_q(n)$ .

Because  $X_q(n)$  satisfies Witten's Conjecture, we can apply Lemma 2.4 to compute the Donaldson invariant,  $D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m}x^m)$ . By (4.10), we have  $0 \notin B'(X_q(n))$ , so  $\nu(K_\varphi) = 1$  (where  $\nu(K)$  is defined in (2.10)) for all  $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$ . If we abbreviate the orientation factor  $\varepsilon(\tilde{w}, K_\varphi)$  in (2.9) by  $\varepsilon(\tilde{w}, \varphi)$ , use the equality  $SW'_{X_q}(K) = SW'_{X_q(n)}(K_\varphi)$  from Theorem 2.1, and note that by (4.7), the set  $B'(X_q(n))$  is enumerated by  $(\mathbb{Z}/2\mathbb{Z})^n$ , then Lemma 2.4 implies that

$$(4.15) \quad \begin{aligned} & D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m}x^m) \\ &= \sum_{\substack{i+2k \\ =\delta-2m}} \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\varepsilon(\tilde{w}, \varphi)} \frac{SW'_{X_q}(K)(\delta-2m)!}{2^{k+n-m}k!i!} \langle K_\varphi, h \rangle^i Q_{X_q(n)}(h)^k. \end{aligned}$$

In addition, the identity (3.6) in Lemma 3.4 gives

$$(4.16) \quad \begin{aligned} & D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m}x^m) \\ &= \sum_{\substack{i+j+2k \\ =\delta-2m}} \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\varepsilon(\tilde{w}, \varphi)} SW'_{X_q}(K) \tilde{b}_{i,j,k}(K_\varphi \cdot \tilde{\Lambda}) \langle K_\varphi, h \rangle^i \langle \tilde{\Lambda}, h \rangle^j Q_{X_q(n)}(h)^k. \end{aligned}$$

We will rewrite (4.15) and (4.16) as sums over terms of the form

$$(4.17) \quad \left\langle K + (-1)^{\pi_1(\varphi)} e_1^*, h \right\rangle^{i_1} \left( \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \right) \langle \Lambda, h \rangle^j Q_{\tilde{X}(n)}(h)^k.$$

Using the definition of  $K_\varphi$  in (4.6), we expand

$$\langle K_\varphi, h \rangle^i = \left\langle (K + (-1)^{\pi_1(\varphi)} e_1^*) + \sum_{u=2}^n (-1)^{\pi_u(\varphi)} e_u^*, h \right\rangle^i,$$

and thus

$$(4.18) \quad \langle K_\varphi, h \rangle^i = \sum_{i_1 + \dots + i_n = i} \binom{i}{i_1, \dots, i_n} (-1)^{\sum_{u=2}^n \pi_u(\varphi) i_u} \left\langle K + (-1)^{\pi_1(\varphi)} e_1^*, h \right\rangle^{i_1} \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u}.$$

Next, we compute the orientation factors for  $\varphi_0 := (0, 0, \dots, 0) \in (\mathbb{Z}/2\mathbb{Z})^n$  to give

$$\begin{aligned} \varepsilon(\tilde{w}, \varphi) &\equiv \frac{1}{2} (\tilde{w}^2 + \tilde{w} \cdot K_\varphi) \pmod{2} \quad (\text{by (2.9)}) \\ &\equiv \frac{1}{2} (\tilde{w}^2 + \tilde{w} \cdot K_0) + \frac{1}{2} (K_\varphi - K_0) \cdot \tilde{w} \pmod{2} \\ &\equiv \varepsilon(\tilde{w}, \varphi_0) + \frac{1}{2} (K_\varphi - K_0) \cdot \tilde{w} \pmod{2} \quad (\text{by (2.9) and (4.6)}) \\ &\equiv \varepsilon(\tilde{w}, \varphi_0) + \sum_{u=1}^n \frac{1}{2} \left( (-1)^{\pi_u(\varphi)} - 1 \right) w_u e_u^* \cdot e_u^* \pmod{2} \quad (\text{by (4.6)}), \end{aligned}$$

and thus, by (4.1),

$$(4.19) \quad \varepsilon(\tilde{w}, \varphi) = \varepsilon(\tilde{w}, \varphi_0) + w_1 \pi_1(\varphi) + \sum_{u=2}^n w_u \pi_u(\varphi) \pmod{2}.$$

Equations (4.18) and (4.19) imply that

$$(4.20) \quad \begin{aligned} &(-1)^{\varepsilon(\tilde{w}, \varphi)} \langle K_\varphi, h \rangle^i \\ &= (-1)^{\varepsilon(\tilde{w}, \varphi_0) + w_1 \pi_1(\varphi)} \sum_{i_1 + \dots + i_n = i} \binom{i}{i_1, \dots, i_n} (-1)^{\sum_{u=2}^n (i_u + w_u) \pi_u(\varphi)} \\ &\quad \times \left\langle K + (-1)^{\pi_1(\varphi)} e_1^*, h \right\rangle^{i_1} \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u}. \end{aligned}$$

We now split the sum in the right-hand-side of (4.15) over  $(\mathbb{Z}/2\mathbb{Z})^n$  into sums over  $\pi_1^{-1}(0)$  and  $\pi_1^{-1}(1)$ :

$$(4.21) \quad \begin{aligned} &D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m} x^m) \\ &= \sum_{\substack{i+2k \\ =\delta-2m}} \frac{SW'_{X_q}(K)(\delta-2m)!}{2^{k+n-m} k! i!} Q_{X_q(n)}(h)^k \\ &\quad \times \left( \sum_{\varphi \in \pi_1^{-1}(0)} (-1)^{\varepsilon(\tilde{w}, \varphi)} \langle K_\varphi, h \rangle^i + \sum_{\varphi \in \pi_1^{-1}(1)} (-1)^{\varepsilon(\tilde{w}, \varphi)} \langle K_\varphi, h \rangle^i \right). \end{aligned}$$



Applying (4.20) to (4.21) yields

$$\begin{aligned}
 (4.22) \quad & D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m}x^m) \\
 &= \sum_{\substack{i_1+\dots+i_n+2k \\ =\delta-2m}} \frac{SW'_{X_q}(K)(\delta-2m)!}{2^{k+n-m}k!i_1!\dots i_n!} (-1)^{\varepsilon(\tilde{w},\varphi_0)} \left( \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \right) Q_{X_q(n)}(h)^k \\
 &\times \left( \sum_{\varphi \in \pi_1^{-1}(0)} (-1)^{\sum_{u=2}^n \pi_u(\varphi)(w_u+i_u)} \langle K + e_1^*, h \rangle^{i_1} \right. \\
 &\quad \left. + (-1)^{w_1} \sum_{\varphi \in \pi_1^{-1}(1)} (-1)^{\sum_{u=2}^n \pi_u(\varphi)(w_u+i_u)} \langle K - e_1^*, h \rangle^{i_1} \right).
 \end{aligned}$$

Identifying  $\pi_1^{-1}(0)$  and  $\pi_1^{-1}(1)$  with  $(\mathbb{Z}/2\mathbb{Z})^{n-1}$  and applying Lemma 4.2 and the definition (4.14) of  $p^{\tilde{w}}(i_2, \dots, i_n)$  yields, for  $a = 0, 1$ ,

$$\sum_{\pi_1^{-1}(a)} (-1)^{\sum_{u=2}^n \pi_u(\varphi)(w_u+i_u)} = p^{\tilde{w}}(i_2, \dots, i_n).$$

Thus, we may rewrite (4.22) as

$$\begin{aligned}
 (4.23) \quad & \frac{(-1)^{\varepsilon(\tilde{w},\varphi_0)}}{SW'_{X_q}(K)} D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m}x^m) \\
 &= \sum_{\substack{i_1+\dots+i_n+2k \\ =\delta-2m}} \frac{(\delta-2m)!}{2^{k+n-m}k!i_1!\dots i_n!} p^{\tilde{w}}(i_2, \dots, i_n) \left( \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \right) Q_{X_q(n)}(h)^k \\
 &\times (\langle K + e_1^*, h \rangle^{i_1} + (-1)^{w_1} \langle K - e_1^*, h \rangle^{i_1}).
 \end{aligned}$$

We next split the sum over  $(\mathbb{Z}/2\mathbb{Z})^n$  on the right-hand-side of (4.16) into sums over  $\pi_1^{-1}(0)$  and  $\pi_1^{-1}(1)$ :

$$\begin{aligned}
 (4.24) \quad & D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m}x^m) = \sum_{\substack{i+j+2k \\ =\delta-2m}} SW'_{X_q}(K) \langle \tilde{\Lambda}, h \rangle^j Q_{X_q(n)}(h)^k \\
 &\times \left( \sum_{\varphi \in \pi_1^{-1}(0)} (-1)^{\varepsilon(\tilde{w},\varphi)} \tilde{b}_{i,j,k}(K_\varphi \cdot \tilde{\Lambda}) \langle K_\varphi, h \rangle^i \right. \\
 &\quad \left. + \sum_{\varphi \in \pi_1^{-1}(1)} (-1)^{\varepsilon(\tilde{w},\varphi)} \tilde{b}_{i,j,k}(K_\varphi \cdot \tilde{\Lambda}) \langle K_\varphi, h \rangle^i \right)
 \end{aligned}$$

We rewrite the argument  $K_\varphi \cdot \Lambda$  in the coefficient  $\tilde{b}_{i,j,k}$ ,

$$\begin{aligned} K_\varphi \cdot \tilde{\Lambda} &= K_0 \cdot \tilde{\Lambda} + (K_\varphi - K_0) \cdot \tilde{\Lambda} \\ &= K_0 \cdot \tilde{\Lambda} + \sum_{u=1}^n ((-1)^{\pi_u(\varphi)} - 1) \lambda_u (e_u^* \cdot e_u^*), \end{aligned}$$

and thus, by (4.1),

$$(4.25) \quad K_\varphi \cdot \tilde{\Lambda} = K_0 \cdot \tilde{\Lambda} + 2\pi_1(\varphi)\lambda_1 + 2 \sum_{u=2}^n \pi_u(\varphi)\lambda_u.$$

Substituting (4.20) and (4.25) into (4.24), together with the definitions (2.9) of  $\varepsilon(\tilde{w}, K_\varphi) \equiv \varepsilon(\tilde{w}, \varphi)$  and (4.6) of  $K_0$ , yields

$$\begin{aligned} (4.26) \quad & \frac{(-1)^{\varepsilon(\tilde{w}, \varphi_0)}}{SW'_{X_q}(K)} D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m} x^m) = \binom{i_1 + \dots + i_n}{i_1, \dots, i_n} \left( \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \right) \langle \tilde{\Lambda}, h \rangle^j Q_{X_q(n)}(h)^k \\ & \times \left( \sum_{\varphi \in \pi_1^{-1}(0)} (-1)^{\theta_\varphi} \tilde{b}_{i,j,k} \left( K_0 \cdot \tilde{\Lambda} + 2 \sum_{u=1}^n \pi_u(\varphi) \lambda_u \right) \langle K + e_1^*, h \rangle^{i_1} \right. \\ & \left. + (-1)^{w_1} \sum_{\varphi \in \pi_1^{-1}(1)} (-1)^{\theta_\varphi} \tilde{b}_{i,j,k} \left( K_0 \cdot \tilde{\Lambda} + 2\lambda_1 + 2 \sum_{u=1}^n \pi_u(\varphi) \lambda_u \right) \langle K - e_1^*, h \rangle^{i_1} \right), \end{aligned}$$

where we write  $\theta_\varphi$  above for

$$\theta_\varphi := \sum_{u=2}^n \pi_u(\varphi)(w_u + i_u).$$

By Lemma 4.2,

$$\begin{aligned} & \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^{n-1}} (-1)^{\sum_{u=2}^n \pi_u(\varphi)(w_u + i_u)} \tilde{b}_{i,j,k} \left( K_0 \cdot \tilde{\Lambda} + 2\pi_1(\varphi)\lambda_1 + 2 \sum_{u=1}^n \pi_u(\varphi)\lambda_u \right) \\ & = \nabla_{2\lambda_2}^{i_2+w_2} \dots \nabla_{2\lambda_n}^{i_n+w_n} \tilde{b}_{i,j,k} \left( K_0 \cdot \tilde{\Lambda} + 2\pi_1(\varphi)\lambda_1 \right). \end{aligned}$$

Substituting the preceding equality into (4.26) yields

$$\begin{aligned} (4.27) \quad & \frac{(-1)^{\varepsilon(\tilde{w}, \varphi_0)}}{SW'_{X_q}(K)} D_{X_q(n)}^{\tilde{w}}(h^{\delta-2m} x^m) \\ & = \sum_{\substack{i_1 + \dots + i_n + j + 2k \\ = \delta - 2m}} \binom{i_1 + \dots + i_n}{i_1, \dots, i_n} \left( \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \right) \langle \tilde{\Lambda}, h \rangle^j Q_{X_q(n)}(h)^k \\ & \times \left( \nabla_{2\lambda_2}^{i_2+w_2} \dots \nabla_{2\lambda_n}^{i_n+w_n} \tilde{b}_{i,j,k}(K_0 \cdot \tilde{\Lambda}) \langle K + e_1^*, h \rangle^{i_1} \right. \\ & \left. + (-1)^{w_1} \left( \nabla_{2\lambda_2}^{i_2+w_2} \dots \nabla_{2\lambda_n}^{i_n+w_n} \tilde{b}_{i,j,k}(K_0 \cdot \tilde{\Lambda} + 2\lambda_1) \langle K - e_1^*, h \rangle^{i_1} \right) \right). \end{aligned}$$

Comparing equations (4.23) and (4.27) gives the desired equality (4.13).  $\square$

We now review a result giving the coefficients  $\tilde{b}_{i,j,k}$  for  $i \geq c(X) - 3$ .

**Proposition 4.7.** [10, Proposition 4.8] *Let  $n > 0$  and  $q \geq 2$  be integers. If  $x, y$  are integers and  $i, j, k, m$  are non-negative integers satisfying, for  $A := i + j + 2k + 2m$ ,*

$$\begin{aligned} (4.28a) \quad & i \geq n, \\ (4.28b) \quad & y > A - 4q - 3 - n, \\ (4.28c) \quad & A \geq 2m, \\ (4.28d) \quad & x \equiv y \equiv 0 \pmod{2}, \end{aligned}$$

*then the coefficients  $\tilde{b}_{i,j,k}(\chi_h, c_1^2, \Lambda \cdot K, \Lambda^2, m)$  defined in (3.4) are given by*

$$\tilde{b}_{i,j,k}(q, q - 3 - n, x, y, m) = \begin{cases} \frac{(A - 2m)!}{k!i!} 2^{m-k-n} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

**Remark 4.8.** The expression for the coefficients  $\tilde{b}_{i,j,k}$  given in Proposition 4.7 differs from that given for the coefficients  $b_{i,j,k}$  in [10, Proposition 4.8] exactly by the factor of  $(-1)$  appearing in the definition (3.4).

Because of the condition (4.28a), Proposition 4.7 only determines the coefficients  $b_{i,j,k}$  with  $i \geq c(X) - 3$ . We next derive a difference equation satisfied by the coefficients  $\tilde{b}_{i,j,k}$  with  $1 \leq i < c(X) - 3$ .

**Proposition 4.9.** *Let  $n > 1$  and  $q \geq 2$  be integers. If  $x, y$  are integers and  $p, j, k, m$  are non-negative integers satisfying, for  $A := p + j + 2k + 2m$ ,*

$$\begin{aligned} (4.29a) \quad & 1 \leq p \leq n - 1, \\ (4.29b) \quad & y > A - 4q - n - 3, \\ (4.29c) \quad & y \equiv A - (n + 3) \pmod{4}, \\ (4.29d) \quad & x - y \equiv 0 \pmod{2}, \end{aligned}$$

*and we abbreviate*

$$\tilde{b}_{p,j,k}(x) = \tilde{b}_{p,j,k}(q, q - n - 3, x, y, m),$$

*then*

$$(4.30) \quad (\nabla_4^1)^{n-p} \tilde{b}_{p,j,k}(x) = 0.$$

*Proof.* Let  $X_q(n)$  be the manifold defined in (4.4). By (4.5), we have  $\chi_h(X_q(n)) = q$  and  $c_1^2(X_q(n)) = q - n - 3$ . We will apply Lemma 4.1 to equation (4.13) for the manifold  $X_q(n)$ . Let  $f_1, f_2 \in H^2(X_q; \mathbb{Z}) \subset H^2(X_q(n); \mathbb{Z})$  and  $K \in B(X_q)$  be the cohomology classes appearing in the properties of  $X_q$  listed at the beginning of Section 4.3, satisfying  $f_i \cdot K = 0$  and  $f_i^2 = 0$  for  $i = 1, 2$  and  $f_1 \cdot f_2 = 1$ . For  $y_0 := \frac{1}{2}(y + (x + 2(n - p))^2 + 4(n - p))$ , define

$$(4.31) \quad \tilde{\Lambda} = \Lambda + \sum_{u=1}^n \lambda_u e_u^*,$$

where we define  $\Lambda := y_0 f_1 + f_2 \in H^2(X_q; \mathbb{Z})$  and the non-negative integers  $\lambda_u$  are given by

$$(4.32) \quad \lambda_u = \begin{cases} -(x + 2(n - p)) & \text{if } u = 1, \\ 0 & \text{if } 1 < u \leq p, \\ 2 & \text{if } p + 1 \leq u \leq n. \end{cases}$$

The assumption (4.29d) that  $y \equiv x \pmod{2}$  implies that  $y_0$  is an integer. Thus, for  $K_0$  as in (4.6), we see that

$$(4.33) \quad \tilde{\Lambda}^2 = y \quad \text{and} \quad \tilde{\Lambda} \cdot K_0 = x.$$

Define  $\tilde{w} := \tilde{\Lambda} - K_0$ , where  $K_0$  is as in (4.6). We claim that  $\tilde{w}, \tilde{\Lambda}$  and  $\delta := A$  satisfy the hypotheses of Lemma 4.6. We have

$$(4.34) \quad \tilde{\Lambda} - \tilde{w} = K_0 \equiv w_2(X_q(n)) \pmod{2}$$

by construction and  $\delta \geq 2m$  by definition.

By (4.31)

$$\begin{aligned} \Lambda^2 &= \tilde{\Lambda}^2 + \sum_{u=1}^n \lambda_u^2 \\ &= y + \sum_{u=1}^n \lambda_u^2 \quad (\text{by (4.33)}) \\ &> \delta - (n + 3) - 4q + \sum_{u=1}^n \lambda_u^2 \quad (\text{by (4.29b) and } \delta = A), \end{aligned}$$

so the condition (4.12a) holds.

For  $\tilde{w} = \tilde{\Lambda} - K_0$  as above, we can write

$$(4.35) \quad \tilde{w} = w + \sum_{u=1}^n w_u e_u^*,$$

where  $w \in H^2(X_q; \mathbb{Z})$ . To verify that the hypothesis (4.12b) in Lemma 4.6 holds, we compute

$$\begin{aligned} \tilde{w}^2 &= \tilde{\Lambda}^2 - 2K_0 \tilde{\Lambda} + K_0^2 \\ &\equiv K_0^2 - \tilde{\Lambda}^2 \pmod{4} \quad (\text{as } -2K_0 \tilde{\Lambda} \equiv -2\tilde{\Lambda}^2 \pmod{4}, \text{ since } K_0 \text{ characteristic}) \\ &\equiv (q - n - 3) - \tilde{\Lambda}^2 \pmod{4} \quad (\text{by (4.5) and (4.9)}) \\ &\equiv (q - n - 3) - \delta + (n + 3) \pmod{4} \quad (\text{by (4.29c) and } \delta = A) \\ &\equiv -\delta - 3q \pmod{4}. \end{aligned}$$

Combining the preceding equality with  $w^2 = \tilde{w}^2 + \sum_{u=1}^n w_u^2$  yields  $w^2 \equiv -\delta - 3q + \sum_{u=1}^n w_u^2 \pmod{4}$  and so condition (4.12b) holds. Hence, we can apply Lemma 4.6 with the given values for  $\tilde{\Lambda}, \tilde{w}, \delta$ , and  $m$  to the coefficients  $\tilde{b}_{p,j,k}(x)$ .

Next, we claim that the set  $\{K + e_1^*, K - e_1^*, e_2^*, \dots, e_n^*, \tilde{\Lambda}, Q_{X_q(n)}\}$  is algebraically independent in the sense of Lemma 4.1. To see that  $K + e_1^*, K - e_1^*, e_2^*, \dots, e_n^*, \tilde{\Lambda}$  are linearly

independent, assume there is a linear combination with  $a, b, c, d_2, \dots, d_n \in \mathbb{R}$ ,

$$(4.36) \quad a(K + e_1^*) + b(K - e_1^*) + c\tilde{\Lambda} + \sum_{u=2}^n d_u e_u^* = 0 \in H^2(X_q(n); \mathbb{R}).$$

Because there is a direct sum decomposition,

$$H^2(X_q(n); \mathbb{R}) \simeq H^2(X_q; \mathbb{R}) \oplus \bigoplus_{u=1}^n \mathbb{R} e_u^*,$$

the equality (4.36) gives

$$(4.37a) \quad (a + b)K + c\Lambda = 0 \in H^2(X_q; \mathbb{R}),$$

$$(4.37b) \quad (a - b)e_1^* + c(\tilde{\Lambda} - \Lambda) + \sum_{u=2}^n d_u e_u^* = 0 \in \bigoplus_{u=1}^n \mathbb{R} e_u^*.$$

By (4.3b), the classes  $K$  and  $\Lambda = y_0 f_1 + f_2$  in  $H^2(X_q; \mathbb{R})$  are linearly independent because  $K, f_1, f_2$  are linearly independent in  $H^2(X_q; \mathbb{R})$ . Thus, (4.37a) implies that  $a + b = 0$  and  $c = 0$ . Equation (4.36) then reduces to

$$a(K + e_1^*) - a(K - e_1^*) + \sum_{u=2}^n d_u e_u^* = 2ae_1^* + \sum_{u=2}^n d_u e_u^* = 0.$$

By the linear independence of  $e_1^*, \dots, e_n^*$ , we have  $a = -b = 0$  and  $d_1 = \dots = d_n = 0$ , proving the linear independence of  $K + e_1^*, K - e_1^*, e_2^*, \dots, e_n^*, \tilde{\Lambda}$ . Next, we observe that the intersection of kernels,

$$\mathbf{K}_1 := \text{Ker}(K + e_1^*) \cap \text{Ker}(K - e_1^*) \cap \text{Ker } \tilde{\Lambda} \cap \bigcap_{u=2}^n \text{Ker } e_u^* \subset H_2(X_q(n); \mathbb{R}),$$

contains the intersection of kernels

$$\mathbf{K}_2 := \text{Ker } K \cap \text{Ker } f_1 \cap \text{Ker } f_2 \subset H_2(X_q; \mathbb{R}).$$

Because the restriction of  $Q_{X_q(n)}$  to  $\mathbf{K}_2$  equals the restriction of  $Q_{X_q}$  to  $\mathbf{K}_2$  and the restriction of  $Q_{X_q}$  to  $\mathbf{K}_2$  is non-zero by (4.3c), the restriction of  $Q_{X_q}$  to  $\mathbf{K}_1$  is also non-zero. Thus, Lemma 4.1 implies that the set  $\{K + e_1^*, K - e_1^*, e_2^*, \dots, e_n^*, \tilde{\Lambda}, Q_{X_q(n)}\}$  is algebraically independent.

This algebraic independence and Lemma 4.1 imply that the coefficients of the term

$$(4.38) \quad \langle K + e_1^*, h \rangle^{i_1} \prod_{u=2}^n \langle e_u^*, h \rangle^{i_u} \langle \Lambda, h \rangle^j Q_{\tilde{X}(n)}(h)^k$$

on the left and right-hand sides of the identity (4.13) in Lemma 4.6 will be equal. In particular, we consider the term (4.38) where

$$(4.39) \quad i_1 = \dots = i_p = 1, \quad i_{p+1} = \dots = i_n = 0.$$

The coefficient of this term on the left-hand-side of (4.13) is given by a multiple of the expression  $p^{\tilde{w}}(i_2, \dots, i_n)$  defined in (4.14). We claim that  $w_n + i_n \equiv 1 \pmod{2}$ , so  $p^{\tilde{w}}(i_2, \dots, i_n) = 0$  and the coefficient of this term vanishes. The equality  $\tilde{\Lambda} - \tilde{w} \equiv w_2(X_q(n)) \pmod{2}$  given

by (4.34) implies  $\lambda_u + w_u \equiv 1 \pmod{2}$  for  $1 \leq u \leq n$ . Combining  $\lambda_u + w_u \equiv 1 \pmod{2}$  for  $1 \leq u \leq n$  with the equality  $\lambda_u \equiv 0 \pmod{2}$  for  $2 \leq u \leq n$  given by (4.32) implies that

$$(4.40) \quad w_u \equiv 1 \pmod{2}, \quad \text{for } 2 \leq u \leq n.$$

In particular,  $w_n \equiv 1 \pmod{2}$  which, combined with  $i_n = 0$  from (4.39), implies that  $w_n + i_n \equiv 1 \pmod{2}$  and so  $p^{\tilde{w}}(i_2, \dots, i_n) = 0$  as asserted. Hence, the coefficient of the term (4.38) on the left-hand-side of (4.13) vanishes. Lemma 4.1 will then imply that the coefficient of the term (4.38) on the right-hand-side of (4.13) vanishes.

By (4.39) and (4.40), the coefficient of the term (4.38) on the right-hand-side of (4.13) is

$$(4.41) \quad p! (\nabla_0^2)^{p-1} (\nabla_4^1)^{n-p} \tilde{b}_{p,j,k}(x) = p! 2^{p-1} (\nabla_4^1)^{n-p} \tilde{b}_{p,j,k}(x).$$

Because Lemma 4.1 implies that the coefficients of the term (4.38) on the left and right-hand sides of (4.13) are equal, the expression given by the right-hand-side of (4.41) must also vanish, giving the desired result.  $\square$

**Remark 4.10.** We required  $p \geq 1$  in Proposition 4.9 because, in order to get information about the coefficients  $\tilde{b}_{0,j,k}$ , we would have to consider the term

$$\langle \Lambda, h \rangle^j Q_{X_q(n)}^k$$

in the equality (4.13). The coefficient of this term on the right-hand-side of (4.13) is a multiple of

$$\nabla_{2\lambda_1}^{w_1} \dots \nabla_{2\lambda_n}^{w_n} \tilde{b}_{0,j,k}(x_0),$$

and so the argument of Proposition 4.9 would show that  $\tilde{b}_{0,j,k}$  also satisfies a difference equation of degree  $n$ . However, the choice of  $\lambda_1$  in (4.32) interacted with the possible values of  $x_0$ , complicating the use of this result. By Lemma 3.5, we can avoid the need to pursue this argument.

Proposition 4.9 and the result for difference equations given by Corollary 4.5 allow us to write the coefficients  $\tilde{b}_{i,j,k}$  as polynomials on  $H_2(X; \mathbb{R})$ . We will combine this fact with Lemma 5.1 to show that, for manifolds of superconformal simple type, the coefficients  $\tilde{b}_{i,j,k}$  with  $i \leq c(X) - 4$  do not contribute to the expression for the Donaldson invariant in (3.6).

**Corollary 4.11.** *Continue the assumptions of Proposition 4.9. In addition assume*

- (1) *There is a class  $K_0 \in B(X)$  such that  $\Lambda \cdot K_0 = 0$ ;*
- (2) *For all  $K \in B(X)$ , we have  $\Lambda \cdot K \equiv 0 \pmod{4}$ .*

*Then for  $1 \leq i \leq n-1$ , the function  $\tilde{b}_{i,j,k}$  is a polynomial of degree  $n-1-i$  in  $\Lambda \cdot K$  and thus*

$$(4.42) \quad \tilde{b}_{i,j,k}(q, q-n-3, K \cdot \Lambda, \Lambda^2, m) = \sum_{u=0}^{n-1-i} \tilde{b}_{u,i,j,k}(q, q-n-3, \Lambda^2, m) \langle K, h_\Lambda \rangle^u,$$

*where  $h_\Lambda = \text{PD}[\Lambda]$  is the Poincaré dual of  $\Lambda$  and if  $u \equiv n+i \pmod{2}$ , then*

$$(4.43) \quad \tilde{b}_{u,i,j,k}(q, q-n-3, \Lambda^2, m) = 0.$$

*Proof.* The Poincaré dual  $h_\Lambda$  has the property that  $\langle K, h_\Lambda \rangle = K \cdot \Lambda$  for any  $K \in H^2(X; \mathbb{Z})$ . The assumption  $\Lambda \cdot K \equiv 0 \pmod{4}$  implies that it is enough to compute  $\tilde{b}_{i,j,k}(q, q - n - 3, 4x, \Lambda^2, m)$  for  $x \in \mathbb{Z}$ . Equation (4.42) then follows from equation (4.30) in Proposition 4.9 and Corollary 4.5. Because  $\Lambda \cdot K \equiv 0 \pmod{4}$ , equation (3.5) implies that

$$\tilde{b}_{i,j,k}(q, q - n - 3, -K \cdot \Lambda, \Lambda^2, m) = (-1)^{n+3+i} \tilde{b}_{i,j,k}(q, q - n - 3, K \cdot \Lambda, \Lambda^2, m).$$

Therefore, the coefficients  $\tilde{b}_{u,i,j,k}$  in (4.42) with  $u \not\equiv n + 3 + i \pmod{2}$ , or equivalently  $u \equiv n + i \pmod{2}$  vanish as asserted in (4.43).  $\square$

**Remark 4.12.** We can remove the assumption in Corollary 4.11 that there is a class  $K_0 \in B(X)$  with  $K_0 \cdot \Lambda = 0$  but then the coefficient will be given as a polynomial in the variable  $\langle K - K_0, h_\Lambda \rangle$  which is less convenient for the computations in the proof of Theorem 1.2.

## 5. PROOFS OF MAIN RESULTS

We begin by establishing the following algebraic consequence of superconformal simple type which will allow us to show that Witten's Conjecture 1.1 holds even without determining the coefficients  $\tilde{b}_{i,j,k}$  with  $i < c(X) - 3$ .

**Lemma 5.1.** *Let  $X$  be a standard four-manifold of superconformal simple type. Assume  $0 \notin B(X)$ . If  $w \in H^2(X, \mathbb{Z})$  is characteristic and  $j, u \in \mathbb{N}$  satisfy  $j + u < c(X) - 3$  and  $j + u \equiv c(X) \pmod{2}$ , then*

$$(5.1) \quad \sum_{K \in B'(X)} (-1)^{\epsilon(w, K)} SW'_X(K) \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u = 0,$$

for any  $h_1, h_2 \in H_2(X; \mathbb{R})$ .

*Proof.* Let  $i = j + u$ . Because  $i \leq c(X) - 4$  by hypothesis, the function  $SW_X^{w,i} : H_2(X; \mathbb{R}) \rightarrow \mathbb{R}$  vanishes identically by the defining property (2.11) of superconformal simple type and thus

$$(5.2) \quad \left. \frac{\partial^i}{\partial s^j \partial t^u} SW_X^{w,i}(sh_1 + th_2) \right|_{s=t=0} = 0.$$

Substituting the equality

$$\begin{aligned} \left. \frac{\partial^i}{\partial s^j \partial t^u} \langle K, sh_1 + th_2 \rangle^i \right|_{s=t=0} &= \frac{\partial^i}{\partial s^j \partial t^u} \sum_{a+b=i} \binom{i}{a} s^a t^b \langle K, h_1 \rangle^a \langle K, h_2 \rangle^b \Big|_{s=t=0} \\ &= i! \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u \end{aligned}$$

into the equality (5.2) and using the expression in (2.11) for  $SW_X^{w,i}$  yields

$$\begin{aligned} 0 &= \frac{\partial^i}{\partial s^j \partial t^u} SW_X^{w,i}(sh_1 + th_2) \Big|_{s=t=0} \\ &= \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \frac{\partial^i}{\partial s^j \partial t^u} \langle K, sh_1 + th_2 \rangle^i \Big|_{s=t=0} \\ &= i! \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u. \end{aligned}$$

This proves that

$$(5.3) \quad 0 = \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u.$$

Because  $SW'_X(K) = (-1)^{\chi_h(X)} SW'_X(-K)$  by [33, Corollary 6.8.4], the terms in (5.3) corresponding to  $K$  and  $-K$ , namely

$$(-1)^{\frac{1}{2}(w^2+w \cdot K)} SW'_X(K) \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u$$

and

$$(-1)^{\frac{1}{2}(w^2-w \cdot K)} SW'_X(-K) \langle -K, h_1 \rangle^j \langle -K, h_2 \rangle^u$$

differ by the sign

$$(-1)^{\chi_h(X) + w \cdot K + j + u}.$$

Because  $w$  is characteristic and because  $X$  has Seiberg-Witten simple type, we have  $w \cdot K \equiv K^2 \equiv c_1^2(X) \pmod{2}$ . Hence,

$$\chi_h(X) + w \cdot K + j + u \equiv \chi_h(X) + c_1^2(X) + j + u \equiv c(X) + j + u \pmod{2}.$$

Hence, the assumptions that  $j + u \equiv c(X) \pmod{2}$  and  $0 \notin B(X)$  imply that the terms in (5.3) corresponding to  $K$  and  $-K$  are equal. Because  $0 \notin B(X)$ ,  $K \neq -K$  for all  $K \in B(X)$  and so by combining these terms, we can rewrite (5.3) as

$$(5.4) \quad 0 = 2 \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u,$$

which yields the desired result.  $\square$

The following allows us to apply Corollary 4.11.

**Lemma 5.2.** *Let  $X$  be a standard four-manifold with odd intersection form. Then for any  $K \in B(X)$ , there is a class  $\Lambda \in H^2(X; \mathbb{Z})$  with  $\Lambda^2 > 0$  and  $\Lambda \cdot K = 0$ .*

*Proof.* Because  $Q_X$  is odd and  $b^+(X) \geq 3$ , by [21, Theorem 1.2.21] we can write

$$H^2(X; \mathbb{Z}) \cong (\oplus_{i=1}^m \mathbb{Z} e_i) \oplus (\oplus_{j=1}^n \mathbb{Z} f_j),$$

where  $m \geq 3$ , and  $Q_X$  is diagonal with respect to the basis  $\{e_1, \dots, e_m, f_1, \dots, f_n\}$ ,  $e_i^2 = 1$ , and  $f_j^2 = -1$ . Because  $K \in B(X)$  is characteristic, we can write

$$K = \sum_{i=1}^m a_i e_i + \sum_{j=1}^n b_j f_j,$$



where  $a_i \equiv 1 \pmod{2}$ . Define  $\Lambda = a_2 e_1 - a_1 e_2$ . Then  $\Lambda \cdot K = 0$  and  $\Lambda^2 = a_1^2 + a_2^2 > 0$  as required.  $\square$

Corollary 4.11 and Lemma 5.1 provide the basis of the proof of our main result:

*Proof of Theorem 1.2.* By Theorem 2.3, we may blow up  $X$  without loss of generality. According to Lemma 2.5, the superconformal simple type condition is preserved under blow-up. If  $\tilde{X}$  is the blow-up of  $X$ , then the characterization of  $B(\tilde{X})$  in (2.4) implies that  $0 \notin B(\tilde{X})$ . Thus, by replacing  $X$  with its blow-up if necessary, we may assume without loss of generality that  $c_1^2(X) \neq 0$ ,  $Q_X$  is odd,  $c(X) \geq 5$ ,  $0 \notin B(X)$  and  $\nu(K) = 1$ , where  $\nu(K)$  is defined in (2.10) for each  $K \in B(X)$ .

By Proposition 2.2, it suffices to prove that equation (2.8) in Lemma 2.4 holds when  $w \in H^2(X; \mathbb{Z})$  is characteristic. Because  $w$  is characteristic,

$$\begin{aligned} w^2 &\equiv \sigma(X) \pmod{8} \quad (\text{by [21, Lemma 1.2.20]}) \\ &= c_1^2(X) - 8\chi_h(X) \quad (\text{by (1.1)}) \\ &\equiv c_1^2(X) \pmod{8}. \end{aligned}$$

Thus,  $D_X^w(h^{\delta-2m}x^m) = 0$  unless  $\delta \equiv -w^2 - 3\chi_h(X) \equiv \chi_h(X) - c_1^2(X) - 4\chi_h(X) \equiv c(X) \pmod{4}$  and we need only compute the Donaldson invariant  $D_X^w(h^{\delta-2m}x^m)$  where

$$(5.5) \quad \delta \geq 2m \quad \text{and} \quad \delta \equiv -w^2 - 3\chi_h(X) \equiv c(X) \pmod{4}.$$

To apply Lemma 3.5 to compute  $D_X^w(h^{\delta-2m}x^m)$ , we abbreviate

$$(5.6) \quad \tilde{b}_{i,j,k}(\Lambda \cdot K) = \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X) - 1, \Lambda \cdot K, \Lambda^2, m),$$

and verify that we can find  $\Lambda \in H^2(X; \mathbb{Z})$  satisfying the conditions of Theorem 3.2 and hence those of Lemma 3.5 as well as of Corollary 4.11.

By Lemma 5.2 and our observation that by replacing  $X$  with its blow-up if necessary we can assume that  $Q_X$  is odd and there are classes  $K_0 \in B(X)$  and  $\Lambda_0 \in H^2(X; \mathbb{Z})$  with  $\Lambda_0^2 > 0$  and  $\Lambda_0 \cdot K_0 = 0$ . Because any  $K \in B(X)$  can be written as  $K = K_0 + 2L_K$  for  $L_K \in H^2(X; \mathbb{Z})$ , if  $\Lambda = 2b\Lambda_0$  where  $b \in \mathbb{N}$ , then

$$(5.7) \quad K_0 \cdot \Lambda = 0 \quad \text{and} \quad K \cdot \Lambda \equiv 0 \pmod{4} \quad \text{for all } K \in B(X),$$

so  $\Lambda$  satisfies two of the assumptions of Corollary 4.11.

If  $w \in H^2(X; \mathbb{Z})$  is characteristic and  $\Lambda = 2b\Lambda_0$ , where  $b \in \mathbb{N}$  and  $\Lambda_0^2 > 0$ , then  $\Lambda - w \equiv w_2(X) \pmod{2}$  and so condition (3.1a) holds. Given  $\delta$ , by picking  $b$  sufficiently large, we can ensure

$$(5.8) \quad \Lambda^2 + c(X) + 4\chi_h(X) > \delta,$$

so condition (3.1b) holds. Conditions (3.1c) and (3.1d) in Theorem 3.2, that  $\delta \equiv -w^2 - 3\chi_h(X) \pmod{4}$  and  $\delta - 2m \geq 0$  respectively, follow from (5.5). Thus, Lemma 3.5 yields

$$(5.9) \quad \begin{aligned} D_X^w(h^{\delta-2m}x^m) &= \sum_{\substack{i+j+2k \\ = \delta-2m}} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} \frac{2(i+1)SW'_X(K)}{\delta - 2m + 1} \tilde{b}_{i+1,j,k}(K \cdot \Lambda) \\ &\quad \times \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k. \end{aligned}$$

We now verify that we can apply Propositions 4.7 and 4.9 and Corollary 4.11 to the coefficients  $\tilde{b}_{i+1,j,k}$  in (5.9). The indices  $i, j, k, m$  appearing in (5.9) satisfy

$$(5.10) \quad i + 1 + j + 2k + 2m = \delta + 1.$$

To match the notation of Propositions 4.7 and 4.9, we will write the first two arguments of the coefficients in (5.6) as

$$(5.11) \quad q := \chi_h(X),$$

and  $c_1^2(X) - 1 = q - 3 - n$ , where

$$(5.12) \quad n := \chi_h(X) - c_1^2(X) - 2 = c(X) - 2.$$

The definitions (5.11) and (5.12), the property that  $b^+ \geq 3$  for standard manifolds, and our earlier observation that we can assume  $c(X) \geq 5$  imply that

$$(5.13) \quad q \geq 2 \quad \text{and} \quad n \geq 2,$$

as required in Propositions 4.7 and 4.9.

We now verify the hypotheses of Proposition 4.7 for the coefficients  $\tilde{b}_{i+1,j,k}$  in (5.6) with  $i \geq c(X) - 3$ . The condition (4.28a) holds because  $i + 1 \geq c(X) - 2 = n$  by (5.12). In the notation of Proposition 4.7 for  $\tilde{b}_{i+1,j,k}$ , we have  $A = i + 1 + j + k + 2m$  and so  $A = \delta + 1$  by (5.10). The property (5.8) of  $\Lambda^2$  and (5.11) imply that

$$(5.14) \quad \Lambda^2 > \delta - c(X) - 4q = \delta - n - 2 - 4q = A - n - 3 - 4q,$$

so condition (4.28b) holds. The condition  $A \geq 2m$  in (4.28c) holds by (5.5). Our choice of  $\Lambda = 2\Lambda_0$  implies that  $\Lambda^2 \equiv \Lambda \cdot K \equiv 0 \pmod{2}$  for all  $K \in B(X)$ , and thus condition (4.28d) holds as well, noting that  $x = \Lambda^2$  and  $y = \Lambda \cdot K$ . Hence, Proposition 4.7 and the equality  $A = \delta + 1$  imply that, for all  $i \geq c(X) - 3$ ,

$$(5.15) \quad \tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) = \begin{cases} \frac{(\delta + 1 - 2m)!}{k!(i + 1)!} 2^{m-k-c(X)+2} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

We now verify the hypotheses of Proposition 4.9 and Corollary 4.11 hold. Observe that  $i + 1 \leq c(X) - 3 = n - 1$  by (5.12), so condition (4.29a) holds. The inequality in (5.14) implies that (4.29b) holds. Because  $A = \delta + 1 \equiv c(X) + 1 \equiv n + 3 \pmod{4}$  by (5.5) and (5.12), the fact that  $\Lambda^2 = (2\Lambda_0)^2 \equiv 0 \pmod{4}$  implies

$$\Lambda^2 \equiv 0 \equiv A - (n + 3) \pmod{4},$$

and thus condition (4.29c) holds. We already showed that condition (4.28d) holds and that implies condition (4.29d) holds. Therefore, Proposition 4.9 applies to the coefficients  $\tilde{b}_{i+1,j,k}$  with  $i \leq c(X) - 3$ . The hypotheses of Corollary 4.11 are those of Proposition 4.9 and the conditions we have previously verified in (5.7). Thus, Corollary 4.11 implies that the coefficients  $\tilde{b}_{i+1,j,k}$  with  $i \leq c(X) - 3$  can be written as

$$(5.16) \quad \begin{aligned} & \tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) \\ &= \sum_{u=0}^{c(X)-4-i} \tilde{b}_{u,i+1,j,k}(q, q - n - 3, \Lambda^2, m) \langle K, h_\Lambda \rangle^u, \end{aligned}$$

where  $h_\Lambda = \text{PD}[\Lambda] \in H_2(X; \mathbb{R})$ .

We now abbreviate,

$$\tilde{b}_{u,i+1,j,k} := \tilde{b}_{u,i+1,j,k}(q, q - n - 3, \Lambda^2, m),$$

and split the sum on the right-hand-side of (5.9) into two parts:

$$\begin{aligned}
 D_X^w(h^{\delta-2m}x^m) &= \sum_{\substack{i+j+2k \\ =\delta-2m, \\ i \leq c(X)-4}} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} \frac{2(i+1)SW'_X(K)}{\delta-2m+1} \\
 &\quad \times \sum_{u=0}^{c(X)-4-i} \tilde{b}_{u,i+1,j,k} \langle K, h \rangle^i \langle K, h_\Lambda \rangle^u \langle \Lambda, h \rangle^j Q_X(h)^k \\
 (5.17) \quad &+ \sum_{\substack{i+j+2k \\ =\delta-2m, \\ i \geq c(X)-3}} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} \frac{2(i+1)SW'_X(K)}{\delta-2m+1} \\
 &\quad \times \tilde{b}_{i+1,j,k}(K \cdot \Lambda) \langle K, h \rangle^i \langle K, h \rangle^j Q_X(h)^k.
 \end{aligned}$$

Because the coefficients  $\tilde{b}_{u,i+1,j,k}$  do not depend on  $\Lambda \cdot K$ , we can rewrite the first sum on the right-hand-side of (5.17) as

$$\begin{aligned}
 (5.18) \quad &\sum_{\substack{i+j+2k \\ =\delta-2m, \\ i \leq c(X)-4}} \frac{2(i+1)SW'_X(K)}{\delta-2m+1} \langle \Lambda, h \rangle^j Q_X(h)^k \\
 &\times \sum_{u=0}^{c(X)-4-i} \tilde{b}_{u,i+1,j,k} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h \rangle^i \langle K, h_\Lambda \rangle^u.
 \end{aligned}$$

By (4.43) and the equality  $n \equiv c(X)$  from (5.12),

$$(5.19) \quad \tilde{b}_{u,i+1,j,k} = 0 \quad \text{if } u \equiv c(X) + i + 1 \pmod{2}.$$

We now consider the terms in the sum (5.18) with  $u \equiv n + i \pmod{2}$ . For  $u$  and  $i$  satisfying  $0 \leq u + i \leq c(X) - 4$ , and  $u \equiv n + i \pmod{2}$ , and  $w \in H^2(X; \mathbb{Z})$  characteristic, Lemma 5.1 implies that

$$\sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h \rangle^i \langle K, h_\Lambda \rangle^u = 0.$$

Because  $0 \leq u \leq c(X) - 4 - i$  and thus  $0 \leq u + i \leq c(X) - 4$  for all terms in the sum (5.18), the preceding equality and (5.19) imply that the sum (5.18) vanishes.

Hence, the terms in the sum (5.17) with  $i \leq c(X) - 4$  vanish. By employing that fact and the formula (5.15) for the coefficients  $\tilde{b}_{i+1,j,k}$ , we can rewrite (5.17) as

$$\begin{aligned}
& D_X^w(h^{\delta-2m}x^m) \\
&= \sum_{\substack{i+2k \\ =\delta-2m, \\ i \geq c(X)-3}} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} \frac{2(i+1)SW'_X(K)}{\delta-2m+1} \\
&\quad \times \frac{(\delta-2m+1)!}{k!(i+1)!2^{k+c(X)-2-m}} \langle K, h \rangle^i Q_X(h)^k \\
&= \sum_{\substack{i+2k \\ =\delta-2m, \\ i \geq c(X)-3}} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \frac{(\delta-2m)!}{k!i!2^{k+c(X)-3-m}} \langle K, h \rangle^i Q_X(h)^k.
\end{aligned}$$

Comparing this equality with (2.8) in Lemma 2.4 and observing that the terms in (2.8) with  $i \leq c(X) - 4$  also vanish by Lemma 5.1, shows that Witten's Conjecture 1.1 holds.  $\square$

**Remark 5.3.** The proof of Theorem 1.2 also illustrates the limits of the method of applying Lemma 4.1 to examples of four-manifolds satisfying Witten's Conjecture 1.1 to determine the coefficients  $\tilde{b}_{i,j,k}$ .

We can see that if  $X$  has superconformal simple type, then by Lemma 5.1, changing the coefficients  $\tilde{b}_{u,i,j,k}$  in (5.17) would not change the expression for the Donaldson invariant given by the cobordism formula because the expression in (5.18) would still vanish. Thus, applying Lemma 4.1 to an equality of the form (5.17), on a manifold of superconformal simple type, does not determine the coefficients  $\tilde{b}_{i,j,k}$ .

Because all standard four-manifolds have superconformal simple type by [7], this makes it unlikely that one could extract more information about the coefficients  $\tilde{b}_{i,j,k}$  by applying this method to other four-manifolds satisfying Witten's Conjecture 1.1.

*Proof of Corollary 1.3.* The result follows immediately from Theorem 1.2 and the result in [7] that standard four-manifolds satisfying the hypotheses of Corollary 1.3 have superconformal simple type.  $\square$

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